The hitchhiker’s guide to the (critical) planar Ising model.

Basic notation:

- $V(G)$ – vertices, $F(G)$ – faces, $\diamondsuit(G)$ – edges, $E(G)$ – oriented edges of $G$;
- $\mathcal{E}(G)$ – set of all even (i.e., degrees of all vertices are even) subgraphs of $G$;
- edge weights $x_e = \exp(-2\beta J_e)$ (or $x_e = \tanh(\beta^* J_e^*)$), parametrization $x_e = \tan(\frac{1}{2} \theta_e)$.

$$Z(G) := \sum_{C \in \mathcal{E}(G)} x(C), \quad x(C) := \prod_{e \in C} x_e.$$  

- Graphs $G^F, G^K, G^C$, double cover $\Upsilon^\times(G)$ and the graph $G^D$: TURN THE PAGE!
- $(x_{[u_1, \ldots, u_m]})_e := -x_e$ on edges crossing the (dual) paths linking faces $u_1, \ldots u_m$ pairwise;
- $\mathcal{E}^{[v_1, \ldots, v_n]}(G)$ – set of all subgraphs of $G$ such that $v_1, \ldots, v_n \in V(G)$ have odd degrees and all other vertices are even;

$$Z_{[v_1, \ldots, v_n]}^{[u_1, \ldots, u_m]}(G) := \sum_{C \in \mathcal{E}^{[v_1, \ldots, v_n]}(G)} x_{[u_1, \ldots, u_m]}(C).$$

Kadanoff–Ceva formalism:

- spins $\sigma_u$ are assigned to faces of $G$, disorders $\mu_v$ – to vertices of $G$.
- real-valued fermions $\chi_c := \mu_v \sigma_u(c)$ are assigned to $\Upsilon(G) = \text{corners of (faces of) } G$;
- Dirac spinor $\eta_c := \varsigma \cdot \exp[-\frac{i}{2} \arg(v(c) - u(c))]$, defined on $c \in \Upsilon^\times(G)$ (where $\varsigma \in \mathbb{C}$ : $|\varsigma| = 1$ is a fixed global prefactor, e.g. $\varsigma = e^{\frac{i\pi}{2}}$ or $\varsigma = 1$);
- (complex)-valued fermions $\psi_c := \eta_c \chi_c$.

- For a collection $\varpi$ of vertices $v_1, \ldots, v_{n-1} \in V(G)$ and faces $u_1, \ldots, u_{m-1} \in F(G)$, let

$$\mathcal{O}_{\varpi}^{[\mu, \sigma]} := \mu_{v_1} \ldots \mu_{v_{n-1}} \sigma_{u_1} \ldots \sigma_{u_{m-1}};$$

$\Upsilon^\times_{\varpi}(G)$ be the double-cover of $\Upsilon(G)$ ramified over each edge, vertex and face of $G$ except over the elements of $\varpi$ and $\Upsilon^\times_{\varpi}(G)$ be the double-cover ramified over $\varpi$.

- The real-valued observables $X_{\varpi}(c) := \mathbb{E}[\chi_c \mathcal{O}_{\varpi}^{[\mu, \sigma]}]$ are spinors on $\Upsilon^\times_{\varpi}(G)$;
- the (complex)-valued observables $\Psi_{\varpi}(c) := \mathbb{E}[\psi_c \mathcal{O}_{\varpi}^{[\mu, \sigma]}] \in \eta_c \mathbb{R}$ are spinors on $\Upsilon^\times_{\varpi}(G)$.

Propagation equation for fermions: (see the TA session)

- $X_{\varpi}(c_2) = X_{\varpi}(c_1) \cdot \cos \theta_e + X_{\varpi}(c_3) \cdot \sin \theta_e$ for each triple of (consecutive on $\Upsilon^\times_{\varpi}(G)$ $][])$ corners surrounding a quad (edge) $e \in \diamondsuit(G)$, where $\theta_e := 2 \arctan x_e$.

- Smirnov’s reformulation for the critical $\mathbb{Z}$-invariant Ising model on isoradial graphs (rhombic lattices): for $z \in \diamondsuit(G)$, there exists $\Psi_{\varpi}(z) \in \mathbb{C}$ such that

$$\Psi_{\varpi}(c) = \frac{1}{2} [\Psi_{\varpi}(z) + \eta_c \cdot \overline{\Psi_{\varpi}(z)}] =: \text{Proj}[\Psi_{\varpi}(z); \eta_c \mathbb{R}].$$

... “But look, you found the notice, didn’t you?”. “Yes,” said Arthur, “yes I did. It was on display in the bottom of a locked filing cabinet stuck in a disused lavatory with a sign on the door saying ‘Beware of the Leopard.’”
Graphs constructed from a given planar graph $G$:

- vertices of $G^F$ ↔ oriented edges and corners of $G$;
- vertices of $G^K$ ↔ oriented edges of $G$;
- vertices of $G^C$ ↔ corners of $G$;
- vertices of $\Upsilon^\times(G)$ ↔ double cover of corners of $G$;
- $G^D$: bipartite, two vertices (of different colors) per corner of $G$.