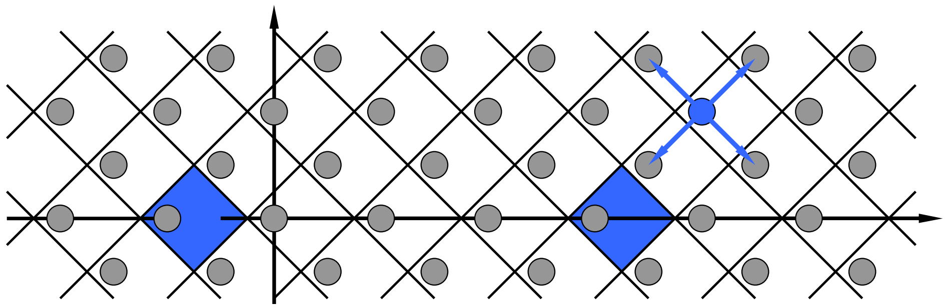
The hitchhiker's guide to the (critical) planar Ising model. TA2.

The goal of this problem set is to compute the limit of the *infinite-volume* 'diagonal' two-point functions $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$ for $x = \tan\frac{1}{2}\theta$, $\theta < \frac{\pi}{4}$:

$$D_{n+1} \;\to\; (1-\tan^4\theta)^{1/4} \quad \text{as} \quad n\to\infty.$$

(this is a version of the famous Onsager–Kaufman–Yang theorem).



We also use the notation $D_n := D_n(x)$, $D_n^* := D_n(x^*)$, where $x^* := \tan\frac{1}{2}(\frac{\pi}{4}-\theta)$. Similarly to the critical point $x_{\text{crit}} = \tan\frac{\pi}{8} = \sqrt{2}-1$, we work with the observable

$$V(k,s) := \langle \chi_{(k,s)}\mu_{(-\frac{1}{2},0)}\sigma_{(2n+\frac{1}{2},0)}\rangle, \quad k,s\in\mathbb{Z},\; k{+}s\in 2\mathbb{Z}.$$

Recall that V satisfies the *massive harmonicity* condition (with $m := \sin 2\theta < 1$):

$$\Delta^{(m)}V(k,s) := \frac{m}{4}\sum_{\pm,\pm} V(k{\pm}1, s{\pm}1) - V(k,s) = 0, \quad (k,s)\neq(0,0),(2n,0).$$

- additional details on how to pass from the three-term propagation equation to the massive harmonicity can be found in [Section 2.4, `arXiv:1904.09168`];
- the computation of D_n at the critical temperature (Wu's formula) can be found in the Appendix of the same paper, see also [Section 3, `arXiv:1605.0903`];

Problem 1. Prove that, for $s\geq 0$,

$$V(k,s) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\frac{k}{2}t}(y(t))^s Q_n(e^{it})dt, \qquad y(t) = \frac{1-(1-m^2\cos^2(\frac{1}{2}t))^{1/2}}{m\cos(\frac{1}{2}t)},$$

where $Q_n(z) = D_n + \ldots + D_n^* z^n$ is a polynomial of degree n with prescribed leading and free terms and such that it is orthogonal to $z,\ldots,z^{n-1}$ with respect to the weight

$$w(e^{it}) := (1{+}q^2)\cdot(1-m^2\cos^2(\tfrac{1}{2}t))^{1/2}, \quad q := \tan\theta < 1,$$

on the unit circle $z = e^{it}$ (note that these properties define Q_n uniquely).

Problem 2 (this is an unpleasant local computation). (a) For $n\geq 1$, prove that

$$w(e^{it})Q_n(e^{it}) = \ldots + D_{n+1} + 0 + q^2 D_{n+1}^* e^{int} + \ldots$$

(b) For $n=0$, argue that the constant term in the Fourier series of $w(e^{it})Q_0(e^{it})$ is $D_1{+}q^2D_1^*$.

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A useful reference: *OPUC on one foot* by Barry Simon, `arXiv:math/0502485`

Let $\Phi_n(z) = z^n + \ldots = \overline{\Phi_n(\overline{z})}$ be the n-th orthogonal polynomial with respect to $w(e^{it})$. Recall the recurrence relation $\Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n \Phi_n^*(z)$, where $\Phi_n^*(z) = z^n \Phi_n(z^{-1})$, and $\alpha_n = \overline{\alpha}_n$ are *Verblunski coefficients*, see Section 2 in the reference quoted above. Recall also that $\beta_n := \|\Phi_n\|^2 = \|\Phi_n^*\|^2 = \beta_0 \prod_{k=0}^{n-1}(1-\alpha_k^2)$, where the norms are taken wrt $\frac{1}{2\pi} w(e^{it})dt$.

Problem 3. **(a)** Prove the recurrence relation

$$\begin{pmatrix} D_{n+1} \\ q^2 D_{n+1}^* \end{pmatrix} = \beta_n \begin{pmatrix} 1 & \alpha_{n-1} \\ \alpha_{n-1} & 1 \end{pmatrix} \begin{pmatrix} D_n \\ D_n^* \end{pmatrix}, \quad n \geq 1.$$

(b) By induction deduce the identity $D_{n+1}\Phi_n^*(q^2) + q^2 D_{n+1}^* \Phi_n(q^2) = \beta_n \ldots \beta_0$.

We now take for granted that $D_n^* = D_n^*(x^*) \leq D_n(x_{\text{crit}}) \to 0$ as $n \to \infty$.

Problem 4. Check that $w(e^{it}) = |1 - q^2 e^{it}|$. Prove that

$$D_{n+1} \;\to\; \frac{\prod_{k=0}^{\infty} \beta_k}{\lim_{n\to\infty} \Phi_n^*(q^2)} \;=\; \frac{(1-q^4)^{-1/4}}{(1-q^4)^{-1/2}} \;=\; (1-q^4)^{1/4} \quad \text{as} \quad n \to \infty$$

due to the Szegö theorems (see Section 8 in the reference quoted above).

For a nice proof of the strong Szegö theorem (the value $\prod_{k=0}^{\infty} \beta_k$) see
A Fredholm determinant formula for Toeplitz determinants
by Alexei Borodin and Andrei Okounkov, `arXiv:math/9907165`