
The hitchhiker's guide to the (critical) planar Ising model. TA3.

Let $\Omega \subset \mathbb{C}$ be a bounded (not necessarily simply connected) domain, $a \in \Omega$ and $|\eta| = 1$. Recall that $f^{[\eta]}(a, \cdot) : \Omega \setminus \{a\} \rightarrow \mathbb{C}$ is defined as the unique[!] *holomorphic* function such that

$$f^{[\eta]}(a, z) = \frac{\bar{\eta}}{z - a} + O(1) \quad \text{as } z \rightarrow a, \quad f^{[\eta]}(a, \zeta) \in (\tau(\zeta))^{-1/2}\mathbb{R}, \quad \zeta \in \partial\Omega,$$

where $\tau(\zeta)$ denotes the tangent vector to Ω (oriented so that Ω remains to the left of $\tau(\zeta)$).

Problem 1. (a) Prove that there exist (unique) functions $f(a, \cdot)$ and $f^*(a, \cdot)$ such that

$$f^{[\eta]}(a, z) = \frac{1}{2}[\bar{\eta}f(a, z) + \eta f^*(a, z)] \quad \text{for all } z \in \Omega \text{ and } |\eta| = 1.$$

(b) Denote $f^{[\eta, \mu]}(w, z) := \text{Re}[\bar{\mu}f^{[\eta]}(w, z)]$, where $w \neq z$ and $|\eta| = |\mu| = 1$. Prove that

$$f^{[\mu, \eta]}(z, w) = -f^{[\eta, \mu]}(w, z).$$

Hint: Consider $\oint_{\partial\Omega} f^{[\eta]}(w, \zeta) f^{[\mu]}(z, \zeta) d\zeta$.

(c) Deduce that $f(z, w) = -f(w, z)$ and $f^*(z, w) = -\overline{f^*(w, z)}$. In particular, $f(z, w)$ is holomorphic in both variables (except at $z = w$) whilst $f^*(w, z)$ is holomorphic in z and anti-holomorphic in w . Argue that the definition

$$\langle \varepsilon_w \rangle_{\Omega}^+ := \frac{i}{2} f^*(w, w)$$

makes sense and that $\langle \varepsilon_w \rangle_{\Omega}^+ \in \mathbb{R}$.

(d) Prove the conformal covariance rules: if $\varphi : \Omega \rightarrow \Omega'$ is a conformal map, then

$$\begin{aligned} f_{\Omega}(w, z) &= f_{\Omega'}(\varphi(w), \varphi(z)) \cdot (\varphi'(w))^{1/2} (\varphi'(z))^{1/2}, \\ f_{\Omega}^*(w, z) &= f_{\Omega'}^*(\varphi(w), \varphi(z)) \cdot \overline{(\varphi'(w))^{1/2}} (\varphi'(z))^{1/2}. \end{aligned}$$

In particular, one has $\langle \varepsilon_w \rangle_{\Omega}^+ = \langle \varepsilon_{\varphi(w)} \rangle_{\Omega'}^+ \cdot |\varphi'(w)|$.

Recall that the *holomorphic spinor* $g_{[v, u]}(z)$ (defined on the double cover of Ω ramified over $u, v \in \Omega$, $u \neq v$) is uniquely characterized by the following conditions:

$$g_{[v, u]}(z) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{z-v}} \cdot [1 + O(z-v)] \quad \text{as } z \rightarrow v; \tag{1}$$

$$g_{[v, u]}(z) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{z-u}} \cdot [c + O(z-u)] \quad \text{as } z \rightarrow u, \quad \text{with an unknown } c \in \mathbb{R}, \tag{2}$$

and the boundary conditions $g_{[v, u]}(\zeta) \in (\tau(\zeta))^{-1/2}\mathbb{R}$ for $\zeta \in \partial\Omega$. Further, recall that $\mathcal{A}(v, u)$ is defined as the next coefficient in the expansion of $g_{[v, u]}(z)$ as $z \rightarrow v$:

$$g_{[v, u]}(z) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{z-v}} \cdot [1 + 2\mathcal{A}(v, u)(z-v) + O((z-v)^2)],$$

and that

$$\langle \sigma_u \sigma_v \rangle_{\Omega}^+ := \exp \left[\int \text{Re}[\mathcal{A}(v, u)dv + \mathcal{A}(u, v)du] \right],$$

where the multiplicative normalization is chosen so that $\langle \sigma_u \sigma_v \rangle_{\Omega}^+ \sim |u-v|^{-1/4}$ as $u \rightarrow v$.

Problem 2. The goal is to prove the *fusion rule* $\sigma\sigma \rightsquigarrow 1 + \frac{1}{2}\varepsilon + \dots$, more precisely:

$$\langle \sigma_v \sigma_u \rangle_{\Omega}^+ = |v - u|^{-1/4} \cdot [1 + \frac{1}{2} \langle \varepsilon_v \rangle_{\Omega}^+ \cdot |v - u| + o(|v - u|)] \quad \text{as } v \rightarrow u \quad (3)$$

(not using explicit expressions available in simply connected Ω), where the correlation functions $\langle \sigma_u \sigma_v \rangle_{\Omega}^+$ and $\langle \varepsilon \rangle_{\Omega}^+$ are defined above.

Denote $\eta := e^{i\frac{\pi}{4}} \cdot (\bar{v} - \bar{u})^{1/2} / |v - u|^{1/2}$. First, take for granted that $\langle \varepsilon_v \rangle_{\Omega}^+ \rightarrow \langle \varepsilon_u \rangle_{\Omega}^+$ and

$$g_{[v,u]}(z) = |v - u|^{1/2} \cdot \left[f^{[\eta]}(v, z) \cdot \left(\frac{z - v}{z - u} \right)^{1/2} + o(1) \right] \quad \text{as } v \rightarrow u, \quad (4)$$

uniformly on compact subsets $z \in \Omega \setminus \{u\}$.

Remark: The right-hand side of (4) is chosen so that the difference does *not* blow up at $z = v$ and approximately satisfies (2) and the boundary conditions, so it should be small.

(a) Deduce from (4) that

$$2\mathcal{A}_{[v,u]} + \frac{1}{2(v - u)} = \langle \varepsilon_v \rangle_{\Omega}^+ \cdot \frac{|v - u|}{v - u} + o(1) \quad \text{as } v \rightarrow u.$$

Hint: Consider $\oint g_{[v,u]}(z) \cdot (z - u)^{1/2} (z - v)^{-3/2} dz$.

(b) Deduce (3) from (4) and the asymptotics $\langle \sigma_u \sigma_v \rangle_{\Omega}^+ \sim |v - u|^{-1/4}$ as $v \rightarrow u$.

(c) Prove that $\langle \varepsilon_v \rangle_{\Omega}^+ \rightarrow \langle \varepsilon_u \rangle_{\Omega}^+$ as $v \rightarrow u$.

Hint: Argue that each subsequential limit of $f^{[\eta]}(v, \cdot)$ must coincide with $f^{[\eta]}(u, \cdot)$.

(d)* Prove (4).

More information on correlations of $\psi, \mu, \sigma, \varepsilon$ and fusion rules: [Section 4, [arXiv:1605.09035](#)]