SUPPLEMENTARY MATERIAL ON SZEGŐ THEOREMS

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1. Let $w : \mathbb{T} \to \mathbb{R}_{\geq 0}$ be a *real symmetric* (i.e., $w(\overline{z}) = w(z)$) weight on the unit circle $\mathbb{T} = \{z \ : \ |z| = 1\}$. The Hilbert space $L^2(\mathbb{T}, \frac{1}{2\pi}w(z)|dz|)$ is defined in a usual way:

$$\langle f, g\rangle_w \ := \ \tfrac{1}{2\pi}\textstyle\int_{\mathbb{T}} f(z)\overline{g(z)}\, w(z)|dz|\,.$$

Orthogonal (with respect to w) polynomials $\Phi_0, \Phi_1, \Phi_2, \ldots$ are defined recursively by

$$\begin{aligned}&\Phi_0(z) = 1,\\ &\Phi_n(z) = z^n + \ldots, \qquad \langle \Phi_n(z), z^m\rangle_w = 0.\end{aligned}$$

In other words, $\Phi_0, \Phi_1, \ldots$ is the result of the orthogonalization procedure applied to the system $1, z, z^2, \ldots$ in $L^2(\mathbb{T}, \frac{1}{2\pi}w(z)|dz|)$. The symmetry condition $w(\overline{z}) = w(z)$ implies that the coefficients of Φ_n are real: $\Phi_n(z) = \overline{\Phi_n(\overline{z})}$.

Set $\Phi_n^*(z) := z^n\Phi_n(z^{-1})$. It is straightforward to check that Φ_n^* is the only polynomial of degree at most n such that $\Phi_n^*(0) = 1$ and $\Phi_n^* \perp z^m$ if $1 \leq m \leq n$.

2. Given $n \geq 0$ consider the polynomial $z\Phi_n(z)$. Properties of Φ_n immediately implies that $z\Phi_n(z)$ is monic and is orthogonal to z^m if $1 \leq m \leq n$. Thus we should have

$$z\Phi_n(z) = \Phi_{n+1}(z) + \alpha_n\Phi_n^*(z). \tag{1}$$

for some constant α_n. Substituting $z = 0$ we find that

$$\alpha_n = -\Phi_{n+1}(0).$$

Coefficients α_n are called *Verblunsky coefficients* and (1) is called *Szegő recurrence relation*. It easily follows from (1) that

$$\|\Phi_n\|_w^2 = \|\Phi_{n+1}\|_w^2 + \alpha_n^2\|\Phi_n\|_w^2$$

and hence

$$\beta_n := \|\Phi_n\|_w^2 = \|\Phi_0\|_w^2 \textstyle\prod_{j=0}^{n-1}(1-|\alpha_j|^2).$$

3. Let us introduce another Hilbert space, the *Hardy space* H^2 in the unit disc:

$$H^2 \ := \ \big\{ f(z) = \textstyle\sum_{j\geq 0} \widehat{f}_j z^j \ \mid \ \|f\|_{H^2}^2 := \sum_{j\geq 0} |\widehat{f}_j|^2 < \infty \big\}. \tag{2}$$

Alternatively one can say that H^2 is the (closed) span of $1, z, z^2, \ldots$ in $L^2(\mathbb{T}, \frac{1}{2\pi}|dz|)$ and one has

$$\|f\|_{H^2}^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |f(z)|^2|dz|.$$

Note that $H^2 \subset L^2(\mathbb{T}, \frac{1}{2\pi}|dz|)$ is exactly the space of square integrable functions on $\mathbb{T}$ that admit holomorphic continuations to the unit disc. Let $P_+ : L^2(\mathbb{T}, \frac{1}{2\pi}|dz|) \to H^2$ be the orthogonal projection (*Riesz projector*).

4. Given a weight w on $\mathbb{T}$ one can define the *Toeplitz operator* with symbol w:

$$T(w) : H^2 \to H^2, \qquad T(w)f := P_+(w \cdot f).$$

If $w = \sum_{j\in\mathbb{Z}} \widehat{w}_j z^j$ is the Fourier series of w, then $T(w)$ can be written as

$$T(w) \ \sim \ \begin{pmatrix} \widehat{w}_0 & \widehat{w}_1 & \widehat{w}_2 & \ddots \\ \widehat{w}_{-1} & \widehat{w}_0 & \widehat{w}_1 & \ddots \\ \widehat{w}_{-2} & \widehat{w}_{-1} & \widehat{w}_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad \text{in the basis } 1, z, z^2, \ldots$$

5. Let $Q_n : H^2 \to H^2$ be the orthogonal projection onto the span Q_nH^2 of $1, z, \ldots, z^n$ in H^2 and $R_n := \mathrm{Id} - Q_n$. The determinants of *finite size* matrices $Q_nT(w)Q_n$ (viewed as operators on the span Q_nH^2) can be easily expressed via the norms β_n of orthogonal polynomials (see paragraph 2):

$$\det_{Q_nH^2} Q_nT(w)Q_n = \beta_0 \cdot \beta_1 \cdot \ldots \cdot \beta_n. \tag{3}$$

To prove (3) just consider the matrix $Q_nT(w)Q_n$ in the basis $\Phi_0, \ldots, \Phi_n$.

6. From now onwards, assume that $w(z) = |D(z)|^2$ where D is holomorphic function in the unit disc and, moreover, an *outer function*. This means that $\mathrm{Re}\log D$ is a harmonic extension of $\frac{1}{2}\log w$ from T to the unit disc and $\mathrm{Im}\log D$ is the harmonic conjugate to this harmonic extension; in particular we assume that $\log w$ is integrable on $\mathbb{T}$.

7. There is a natural way to interpret Φ_n^* as a solution to some extremal problem. Namely, let us consider a problem of minimizing

$$\frac{1}{2\pi}\int_{\mathbb{T}} |\Phi(z)|^2\, w(z)|dz| \tag{4}$$

over all polynomials Φ such that $\deg\Phi \le n$ and $\Phi(0) = 1$. One can think of this minimum as of the distance between 0 and the affine hyperplane in Q_nH^2 defined by the condition $\Phi(0) = 1$. Clearly, this distance is achieved on the unique vector which is orthogonal to $z, z^2, \ldots, z^n$, the polynomial Φ_n^*.

8. If we drop the condition that Φ is a polynomial, then the minimum of (4) is attained on the function $\Phi(z) := D(0)/D(z)$ (recall that we have $w(z) = |D(z)|^2$ and the outer function $D(z)$ does not vanish in the unit disc). Indeed, a straightforward computation shows that $D(0)/D(z)$ is orthogonal to $z, z^2, \ldots$:

$$\frac{1}{2\pi}\int_{\mathbb{T}} (D(z))^{-1}\overline{z}^k w(z)|dz| \ = \ \frac{i}{2\pi}\int_{\mathbb{T}} \overline{D(z)z^{k-1}dz} \ = \ 0.$$

This leads to the following result:

Theorem 1 (first Szegő theorem). *In the setup described above one has*

(1) $\beta_n = \|\Phi_n^*\|_w^2 \to |D(0)|^2$ *as* $n \to \infty$*;*
(2) $\Phi_n^*(z) \to D(0)/D(z)$ *as* $n \to \infty$*, uniformly on compact subsets of the unit disc.*

9. Assume now that $D(0)=1$ (this can be achieved just by a renormalization of the weight). Then $\beta_n\to 1$ and it is reasonable to ask if one can compute the *Fredholom determinant* $\det T(z)=\lim_{n\to\infty}\det_{Q_nH^2}Q_nT(w)Q_n=\prod_{j=0}^{+\infty}\beta_j$ of the Toeplitz operator $T(w)$, see (3). The answer to this question is provided by

Theorem 2 (second (or strong) Szegő theorem). *Assume that* $\frac{d}{dz}\log D$ *is square integrable on the unit disc and let* $\log D(z)=\sum_{k\ge 0}L_kz^k$. *Then we have*

$$\prod_{j=0}^{+\infty}\beta_j \;=\; \det T(w) \;=\; \exp\Big[\sum_{k\ge 1}k|L_k|^2\Big] \;=\; \exp\Big[\frac{1}{\pi}\int_{\{z\,:\,|z|<1\}}\Big|\frac{D'(z)}{D(z)}\Big|^2 dA(z)\Big].$$

The aim of the next paragraphs is to sketch the proof of this theorem that is due to Borodin and Okunkov. The fact that the two last quantities match is a straightforward computation, thus it remains to show that $\det T(w)=\exp\big[\sum_{k\ge1}k|L_k|^2\big]$.

10. Given a (say, bounded) function $a:\mathbb{T}\to\mathbb{C}$, let us introduce the *Hankel operator*

$$H(a):H^2\to H^2,\qquad (H(a)f)(z):=P_+(\overline{z}a(z)\cdot f(\overline{z})). \tag{5}$$

If $a=\sum_{j\in\mathbb{Z}}\widehat{a}_jz^j$, then

$$H(a)\;\sim\;\begin{pmatrix}\widehat{a}_1 & \widehat{a}_2 & \widehat{a}_3 & \cdot^{\cdot^{\cdot}}\\ \widehat{a}_2 & \widehat{a}_3 & \widehat{a}_4 & \cdot^{\cdot^{\cdot}}\\ \widehat{a}_3 & \widehat{a}_4 & \widehat{a}_5 & \cdot^{\cdot^{\cdot}}\\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}}\end{pmatrix}\quad \text{in the basis } 1,z,z^2,\ldots$$

For shortness, denote $\widetilde{a}(z):=a(\overline{z})$, $z\in\mathbb{T}$. It is easy to see that

$$a(z)f(z)=(T(a)f)(z)+\overline{z}\,(H(\widetilde{a})f)(\overline{z}),$$

which implies the following identity:

$$T(a_1a_2)=T(a_1)T(a_2)+H(a_1)H(\widetilde{a}_2). \tag{6}$$

Using this relation we conclude that, if a_1 is anti-holomorphic *or* a_2 is holomorphic in the unit disc, then

$$T(a_1a_2)=T(a_1)T(a_2). \tag{7}$$

In particular, in these cases one has $T(a_1^{-1})=T(a_1)^{-1}$ and $T(a_2^{-1})=T(a_2)^{-1}$.

11. The key trick. Due to (7) we have $T(w)=T(\overline{D}D)=T(\overline{D})T(D)$. It follows that

$$\begin{aligned} T(D^{-1})T(w)T(\overline{D}^{-1}) \;&=\; \big(T(D^{-1})T(\overline{D})\big)\big(T(D)T(\overline{D}^{-1})\big)\\ &=\; T(\overline{b})^{-1}T(b)^{-1} \;=\; \big(T(b)T(\overline{b})\big)^{-1}, \end{aligned}\tag{8}$$

where $b:=\overline{D}^{-1}D$. Note that both operators $T(D^{-1})$ and $T(\overline{D}^{-1})$ are triangular in the basis $1,z,z^2,\ldots$ with diagonal entries $D(0)^{-1}=\overline{D}^{-1}(0)=1$. Therefore,

$$\begin{aligned} \det_{Q_nH^2}Q_nT(w)Q_n \;&=\; \det_{Q_nH^2}Q_nT(D^{-1})T(w)T(\overline{D}^{-1})Q_n\\ &=\; \det_{Q_nH^2}Q_n\big(T(b)T(\overline{b})\big)^{-1}Q_n. \end{aligned}\tag{9}$$

Since $b\bar{b} = 1$, the identity (6) implies

$$T(b)T(\bar{b}) = \mathrm{Id} - H(b)H(b)^*. \tag{10}$$

Provided the Fourier coefficients of the function $b = \overline{D}^{-1}D$ decay fast enough (which is always the case if D is smooth enough and does not vanish on $\mathbb{T}$), the Hankel operator $H(b)$ is Hilbert–Schmidt and hence $H(b)H(b)^*$ is a trace class operator. In this situation, the Jacobi identity allows one to rewrite (9) in the following form:

$$\det_{Q_nH^2} Q_nT(w)Q_n \;=\; \frac{\det_{R_nH^2}(\mathrm{Id} - R_nH(b)H(b)^*R_n)}{\det(\mathrm{Id} - H(b)H(b)^*)}\,,$$

recall that $R_n = \mathrm{Id} - Q_n$. Passing to the limit $n \to \infty$, one gets

$$\det T(w) = \lim_{n\to\infty} \det_{Q_nH^2} Q_nT(w)Q_n \;=\; (\det(\mathrm{Id} - H(b)H(b)^*))^{-1}. \tag{11}$$

12. To compute the last determinant we use the *Helton-Howe formula*:

Theorem 3. *Let A, B be bounded operators on a Hilbert space such that $[A, B]$ is a trace class operator. Then $e^Ae^Be^{-A}e^{-B} - \mathrm{Id}$ is a trace class operator and*

$$\det e^Ae^Be^{-A}e^{-B} = \exp(\mathrm{Tr}[A, B]).$$

It is easy to see that

$$\mathrm{Id} - H(b)H(b)^* \;=\; T(b)T(\bar{b}) \;=\; T(\overline{D}^{-1})T(D)T(\overline{D})T(D^{-1})$$

and (7) implies that $T(D) = e^{T(\log D)}$ and $T(\overline{D}) = e^{T(\log \overline{D})}$. Therefore,

$$\det(\mathrm{Id} - H(b)H(b)^*) = \exp(-\mathrm{Tr}[T(\log \overline{D}), T(\log D)]).$$

Due to (7) and (10) we also have

$$[T(\log \overline{D}), T(\log D)] = \mathrm{Id} - T(\log D)T(\log \overline{D}) = H(\log D)H(\log D)^*.$$

Finally, a straightforward computation implies that

$$\mathrm{Tr}(H(\log D)H(\log D)^*) = \textstyle\sum_{k\geq 0} k|L_k|^2$$

and we are done.

References:

- Alexei Borodin and Andrei Okounkov. A Fredholm determinant formula for Toeplitz determinants. `arXiv:math/9907165`
- Barry Simon, OPUC on one foot. `arXiv:math/0502485`
- Barry Simon, Orthogonal Polynomials on the Unit Circle. AMS Colloquium Publications, Volume 54.
- Percy Deift, Alexander Its, and Igor Krasovsky. Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model. Some history and some recent results. `arXiv:arXiv:1207.4990`