The hitchhiker's guide to the (critical) planar Ising model. TA1. Solutions.

Problem 1 (Kasteleyn's theorem). Recall that, for an antisymmetric $(2n)\times(2n)$ matrix A, the *Pfaffian* of A is defined as

$$\mathrm{Pf}[A] \ := \ (2^n n!)^{-1}\textstyle\sum_{\pi\in S_{2n}}(-1)^{\mathrm{sign}(\pi)}a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}.$$

(a) Prove the identity $(\mathrm{Pf}[\,A\,])^2=|\det A|$.

Recall that a Kasteleyn orientation of edges of a planar graph is defined by the property that each face has odd number of edges oriented clockwise, and let $A=-A^\top$ be a signed (according to such an orientation) adjacency matrix of a finite planar graph.
(b) Prove the Kasteleyn theorem:

$$\mathcal{Z}_{\mathrm{dimers}}(G) \ := \ (2^n n!)^{-1}\textstyle\sum_{\pi\in S_{2n}}|a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}| \ = \ |\,\mathrm{Pf}[A]\,|.$$

Solution. (a). Let us start with an "linear algebra" solution. Given an antisymmetric matrix A one can consider the 2-form

$$\omega=\sum_{i,j=1}^{2n}A_{ij}dx_i\wedge dx_j.$$

Since A is antisymmetric, this form is correctly defined and one can see from the definition of the Pfaffian that

$$\omega^n=\mathrm{Pf}[A]\,dx_1\wedge\dots\wedge dx_n.$$

Using this observation one concludes that $\mathrm{Pf}[U^TAU]=(\det U)\cdot\mathrm{Pf}[A]$ for any matrix U. Recall that any antisymmetric matrix is of the form

$$U^T\begin{pmatrix}0 & P\\ -P^T & 0\end{pmatrix}U$$

for some *orthogonal* matrix U with determinant equal to 1. Computing Pfaffians of both sides of this identity we get

$$\mathrm{Pf}[A]=\mathrm{Pf}\left[\begin{pmatrix}0 & P\\ -P^T & 0\end{pmatrix}\right]=\det P=\pm\sqrt{\det A}.$$

Another way to solve the problem is to play with combinatorics of perfect matchings.

Step 1. Let K_{2n} be the complete graph on $2n$ vertices labeled by $1,2,\dots,2n$ and D be a perfect matching of K_{2n}. Given a permutation $\pi\in S_{2n}$ let us write $\pi\sim D$ if the perfect matching D is given by edges $(\pi(1),\pi(2)),\dots,(\pi(2n-1),\pi(2n))$. Define

$$wt(D):=(-1)^{\mathrm{sign}(\pi)}a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}$$

where $\pi\in D$. Assume that $\pi'\in D$. Then π' differs from π by k transpositions, so $(-1)^{\mathrm{sign}(\pi)}=(-1)^k\cdot(-1)^{\mathrm{sign}(\pi')}$, but, on the other hand, $a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}=(-1)^k a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}$ due to the fact that A is antisymmetric. It follows that $wt(D)$ does not depend on the choice of π. It is easy to see that the number of permutations that correspond a given perfect matching is equal to $n!2^n$. We conclude that

$$\mathrm{Pf}[A] = \sum_{\substack{D \text{ – is perfect} \\ \text{matching for } K_{2n}}} wt(D).$$

Step 2. If D_1, D_2 are two perfect matchings then $D_1 \cup D_2$ defines a decomposition of K_{2n} into a disjoint union of even cycles and edges (an edge occurs in this decomposition if it belongs to both D_1 and D_2). Let $\pi \in S_{2n}$ and $\pi = s_1 \circ \cdots \circ s_k$ is its cyclic decomposition. Let us write that $\pi \sim D_1 \cup D_2$ if s_j's corresponds to the cycles from $D_1 \cup D_2$ defined (edges are considered as cycles of length two, i.e. as transpositions). Note that if $\pi \sim D_1 \cup D_2$ then one can always find $\pi_1 \sim D_1$ and $\pi_2 \sim D_2$ such that $\pi = \pi_1 \circ \pi_2^{-1}$. In particular, $(-1)^{\mathrm{sign}(\pi)} = (-1)^{\mathrm{sign}(\pi_1)} \cdot (-1)^{\mathrm{sign}(\pi_2)}$ and we have

$$(-1)^{\mathrm{sign}(\pi)} a_{1,\pi(1)} \dots a_{2n,\pi(2n)} = wt(D_1) \cdot wt(D_2).$$

Notice that π corresponds to $D_1 \cup D_2$ for some D_1, D_2 if and only if all cycles of π are even. Let us call those permutations *even*. Using the observation above one can easily check that

$$\sum_{\pi \text{ – even}} (-1)^{\pi} \prod_{j=1}^{2n} a_{j\pi(j)} = \sum_{\substack{D_1, D_2 \text{ – perfect} \\ \text{matchings for } K_{2n}}} wt(D_1) \cdot wt(D_2) = \mathrm{Pf}[A]^2.$$

Step 3. Write

$$\det A = \sum_{\pi \text{ – even}} (-1)^{\pi} \prod_{j=1}^{2n} a_{j\pi(j)} + \sum_{\pi \text{ – not even}} (-1)^{\pi} \prod_{j=1}^{2n} a_{j\pi(j)}.$$

Due to the previous step it remains to show that the second summand is this expression is zero. Assume that π is not even. If π has a fixed point, say, $\pi(j) = j$, then $\prod_{j=1}^{2n} a_{j\pi(j)} = 0$ since $a_{j,j} = 0$ because A is antisymmetric. Now let $\pi = s_1 s_2 \dots s_k$ where s_j is a cyclic permutation and assume that s_1 is an odd cycle of length greater then 1. Then let $\pi' = s_1^{-1} s_2 \dots s_k$. Then the fact that A is antisymmetric implies that

$$(-1)^{\mathrm{sign}(\pi)} \prod_{j=1}^{2n} a_{j\pi(j)} = -(-1)^{\mathrm{sign}(\pi')} \prod_{j=1}^{2n} a_{j\pi'(j)}.$$

Using this observation one can easily prove that all non-zero summands in the sum

$$\sum_{\pi \text{ – not even}} (-1)^{\pi} \prod_{j=1}^{2n} a_{j\pi(j)}$$

cancels out.

(b). Note that if A is the adjacency matrix then $a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}$ is non-zero only if the pair of vertices $(\pi(2n-1),\pi(2n))$ form an edge in the graph, so we find that

$$\begin{aligned}\mathrm{Pf}[A] &= (2^n n!)^{-1}\sum_{\pi\in S_{2n}}(-1)^{\mathrm{sign}(\pi)}a_{\pi(1)\pi(2)}\dots a_{\pi(2n-1)\pi(2n)}\\ &= \sum_{\substack{D\text{ is perfect}\\ \text{matching}}} wt(D).\end{aligned}$$

What we need to show is that if A is a Kasteleyn matrix, then the sing of $wt(D)$ is the same for any perfect matching D.

Let D_1 and D_2 be two perfect matchings and let $\pi_1\sim D_1$, $\pi_2\sim D_2$ be chosen such that cycles of the permutation $\pi:=\pi_1\circ\pi_2^{-1}$ correspond to the cycles of the decomposition given by $D_1\cup D_2$. Let us write $\pi=s_1\circ\dots\circ s_k$ where s_j's are corresponding cycle permutations. Since s_j's are even we get that $(-1)^\pi=(-1)^k$. Is follows that (cf. *Step 2* above)

$$wt(D_1)\cdot wt(D_2)=\prod_{j=1}^k\left(-\prod_{(u,v)\in s_j}a_{u,v}\right)$$

where $(u,v)\in s_j$ if $s(u)=v$ and $u\neq v$. Thus, to show that $wt(D_1)\cdot wt(D_2)$ is positive we need to show that $\prod_{(u,v)\in s_j}a_{u,v}$ is negative for any j. From now on let us consider s_j as an oriented cycle on the graph. Then $(u,v)\in s_j$ if the corresponding oriented edge belongs to this cycle. Denote by $\overline{e}(s_j)$ the number of edges from s_j such that the orientation of s_j does not coincide with the Kasteleyn orientation them. We have

$$\mathrm{sign}\left(\prod_{(u,v)\in s_j}a_{u,v}\right)=(-1)^{\overline{e}(s_j)}.$$

Let us show that $\overline{e}(s_j)$ is odd for any j. Let G_j be the subgraph of G that consists of all vertices that lie *inside* s_j or belong to s_j (here we use the planar structure of G!!). Since all the cycles s_j are non-intersecting we find that the union of s_j with all the cycles s_i that lies inside s_j covers all vertices of G_j, thus the number of vertices of G_j is even. Applying the Euler formula to G_j we find that

$$\#E(G_j)+1=\#F(G_j)\mod 2.$$

Recall that for each face $f\in F(G_j)$ there is odd number of edges oriented clockwise with respect to this edge. Denote the number of such edges by $\overline{e}(f)$. The following identities hold in $\mathbb{Z}/2\mathbb{Z}$:

$$\begin{aligned}&\#E(G_j)+1\equiv\#F(G_j)\equiv\sum_{f\in F}\overline{e}(f)\equiv\\ &\equiv\#\{\text{edges lying inside } s_j\}+\mathrm{length}(s_j)+\overline{e}(s_j)=E(G_j)+\overline{e}(s_j)\end{aligned}$$

where we use that $\mathrm{length}(s_j)$ is even. It follows that $\overline{e}(s_j)=1\mod 2$.

Problem 2 (Kramers–Wannier duality for spins and disorders). Recall that $\mu_{v_1}\dots\mu_{v_n}$ can be viewed as a random variable $\exp[-2\beta\sum_{e=(uw):e\cap\gamma^{[v_1,\dots,v_n]}\neq\emptyset}J_e\sigma_u\sigma_w]$, where $\gamma^{[v_1,\dots,v_n]}$ is the union of disorder paths linking the vertices $v_1,\dots,v_n\in V(G)$ pairwise (in particular, we impose that n is even). Let $u_1, \dots, u_m$ be a collection of faces and let $\gamma_{[u_1,\dots,u_m]}$ be a union of paths on the dual graph linking the faces $u_1, \dots, u_m$ pairwise (if m is odd then one of u_j's is supposed to be linked with the outer face). Recall that given a collection of paths $\gamma_{[u_1,\dots,u_m]}$ we can define $Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)$. Set $(-1)^d := (-1)^{\gamma^{[v_1,\dots,v_n]}\cdot\gamma_{[u_1,\dots,u_m]}}$ where $\gamma^{[v_1,\dots,v_n]} \cdot \gamma_{[u_1,\dots,u_m]}$ is the total number of intersections between paths modulo 2.

(a) Argue that $\mathbb{E}[\mu_{v_1} \dots \mu_{v_n} \sigma_{u_1} \dots \sigma_{u_m}] = (-1)^d Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G) \cdot (Z(G))^{-1}$.

(b) Using the high-temperature expansion of the dual Ising model on the double-cover branching over $u_1, \dots, u_m$ prove that $\mathbb{E}^*[\sigma^*_{v_1} \dots \sigma^*_{v_n} \mu^*_{u_1} \dots \mu^*_{u_m}] = \pm Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G) \cdot (Z(G))^{-1}$, where $\mu^*_{u_1} \dots \mu^*_{u_m}$ can be defined similarly to $\mu_{v_1} \dots \mu_{v_n}$ by choosing paths linking $u_1, \dots u_m$ (and possibly u_{out}) on the dual graph.

Solution. **(a)** Recall the definition of $Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)$: set $s(e) = 1$ if $e \cap \gamma_{[u_1,\dots,u_m]} = \varnothing$ and $s(e) = -1$ in the opposite case and write

$$Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G) := \sum_{C \in \mathcal{E}(G; v_1,\dots,v_n)} x_{[u_1,\dots,u_m]}(C)$$

where $(x_{[u_1,\dots,u_m]})_e = s(e)x_e$ and $s(C) = \prod_{e \in C} s(e)$.

Step 1. Given an even subgraph $C \subset \mathcal{E}(G)$ let us denote by $C\Delta\gamma^{[v_1,\dots,v_n]}$ the symmetric difference (i.e. $e \in C\Delta\gamma^{[v_1,\dots,v_n]}$ iff e belongs only to one of C and $\gamma^{[v_1,\dots,v_n]}$). Observe that $C \mapsto C\Delta\gamma^{[v_1,\dots,v_n]}$ is a bijection between $\mathcal{E}(G)$ and $\mathcal{E}(G; v_1, \dots, v_n)$. We have

$$s(C\Delta\gamma^{[v_1,\dots,v_n]}) = (-1)^d \cdot s(C).$$

Step 2. Let $\sigma = \{\sigma_u\}_{u \in F(G)}$ be a spin configuration and let $C \in \mathcal{E}(G)$ be the corresponding domain wall. Observe that

$$\left(\prod_{j=1}^m \sigma_{u_j}\right) \exp\left(\beta \sum_{e=(w,w')} J_e \sigma_w \sigma_{w'}\right) = \exp\left(\beta \sum_{e=(w,w')} J_e\right) \cdot x_{[u_1,\dots,u_m]}(C).$$

Step 3. Let $\sigma = \{\sigma_u\}_{u \in F(G)}$ be a spin configuration and let $C \in \mathcal{E}(G)$ be the corresponding domain wall. Using precious two steps observe that

$$\left(\prod_{(ww')\cap\gamma^{[v_1,\dots,v_n]}\neq\emptyset} x_e^{\sigma_w\sigma_{w'}}\right)\left(\prod_{j=1}^m \sigma_{u_j}\right) \exp(\beta \sum_{e=(w,w')} J_e \sigma_w \sigma_{w'}) =$$
$$= (-1)^d \exp\left(\beta \sum_{e=(w,w')} J_e\right) \cdot x_{[u_1,\dots,u_m]}(C\Delta\gamma^{[v_1,\dots,v_n]}).$$

Step 4. Using previous steps write

$$\mathbb{E}[\mu_{v_1} \dots \mu_{v_n} \sigma_{u_1} \dots \sigma_{u_m}] = \mathcal{Z}^{-1} \sum_\sigma \left(\prod_{(ww')\cap\gamma^{[v_1,\dots,v_n]}\neq\emptyset} x_e^{\sigma_w\sigma_{w'}}\right)\left(\prod_{j=1}^m \sigma_{u_j}\right) \exp(\beta \sum_{e=(w,w')} J_e \sigma_w \sigma_{w'})$$
$$= (-1)^d \cdot Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G)/Z(G)$$

(b) Recall that we have fixed paths $\gamma^{[v_1,\dots,v_n]}$ and $\gamma_{[u_1,\dots,u_m]}$ such that $\gamma^{[v_1,\dots,v_n]}\cdot\gamma_{[u_1,\dots,u_m]}=0$. Let us expand $\mathbb{E}^*[\sigma^*_{v_1}\dots\sigma^*_{v_n}\mu^*_{u_1}\dots\mu^*_{u_m}]$ via the high-temperature expansion. Using the definition of the random variable $\mu^*_{u_1}\dots\mu^*_{u_m}$ we can write:

$$\begin{aligned}
\mathbb{E}^*[\sigma^*_{v_1}\dots\sigma^*_{v_n}\mu^*_{u_1}\dots\mu^*_{u_m}] &= \mathcal{Z}^{-1}\left(\sum_{\sigma^*}\prod_{j=1}^n\sigma^*_{v_j}\prod_{(vv')\cap\gamma_{[u_1,\dots,u_n]}\neq\emptyset}(x^*_e)^{\sigma^*_v\sigma^*_{v'}}\right)\cdot\exp\left(\beta^*\sum_{e=(v,v')}J^*_e\sigma^*_v\sigma^*_{v'}\right)\\
&= \mathcal{Z}^{-1}\sum_{\sigma^*}\left(\prod_{j=1}^n\sigma^*_{v_j}\right)\cdot\exp\left(\beta^*\sum_{e=(v,v')}J^*_e s(e)\sigma^*_v\sigma^*_{v'}\right)\\
&= \mathcal{Z}^{-1}\sum_{\sigma^*}\left(\prod_{j=1}^n\sigma^*_{v_j}\right)\cdot\prod_{e=(v,v')}(\cosh(\beta^*J^*_e)+\sinh(\beta^*J^*_e)s(e)\sigma^*_v\sigma^*_{v'})\\
&= \mathcal{Z}^{-1}\left(\prod_e\cosh(\beta^*J^*_e)\right)\sum_{\sigma^*}\prod_{j=1}^n\sigma^*_{v_j}\prod_{e=(v,v')}(1+\tanh(\beta^*J^*_e)s(e)\sigma^*_v\sigma^*_{v'})\\
&= Z(G)^{-1}\sum_{C\in\mathcal{E}(G;v_1,\dots,v_n)}x(C)s(C)\\
&= Z(G)^{-1}\sum_{C\in\mathcal{E}(G;v_1,\dots,v_n)}x_{[u_1,\dots,u_m]}(C)\\
&= Z(G)^{-1}\cdot Z^{[v_1,\dots,v_n]}_{[u_1,\dots,u_m]}(G).
\end{aligned}$$

Problem 3 (anti-commutativity of variables $\psi_c=\eta_c\mu_{v(c)}\sigma_{u(c)}$). Recall that the spin-disorder correlations $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ are defined up to a sign which has the same branching structure as $\left[\prod_{p=1}^2\prod_{q=1}^2(v_p-u_q)\right]^{1/2}$. Argue that $\mathbb{E}[\psi_c\psi_d]$, $c\neq d$ (and, more generally, $\mathbb{E}[\psi_c\psi_d\mathcal{O}^{[\mu,\sigma]}_\varpi]$) is a *function* of $(c,d)\in\Upsilon(G)\times\Upsilon(G)\setminus\{(c,c),c\in\Upsilon(G)\}$ (resp., $c,d\in\Upsilon_\varpi(G)$) and that this function is anti-symmetric: $\mathbb{E}[\psi_d\psi_c]=-\mathbb{E}[\psi_c\psi_d]$, $c\neq d$.

Solution. Currently (after Problem 2), we have two ways to define a spin-disorder correlation $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$: one is to fix a path $\gamma^{[v_1,v_2]}$, then to interpret $\mu_{v_1}\mu_{v_2}$ as a random variable and $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ as a literal expectation. Another way is to fix a path $\gamma_{[u_1,u_2]}$ and to set $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]:=Z^{[v_1,v_2]}_{[u_1,u_2]}(G)\cdot Z(G)^{-1}$; as we have seen in Problem 2(a) these two ways agree up to a sign. Note that in the first case we get a function that branches when v_p makes a turn around u_q (and u_q is supposed to be fixed) and in the second case we get a function that branches other way around: when u_q make a turn around v_p (and v_p is fixed). From this heuristics we see that the first definition is good if one wants to consider $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ as a function of v_1,v_2 and the second works well if one consider $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ as a function of u_1,u_2. If we want to consider $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ as a function of v_1,v_2,u_1,u_2, then we should "superimpose" these two definitions. One way is to consider $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ as an spin-spin correlation of a spin-flip model on the double cover branched at v_1,v_2. But let us present an explicit combinatorial construction instead: hopefully, it will make the situation clear for those who are not familiar with the construction of a branched cover. Start from some "base point" $v_1^{(0)},v_2^{(0)},u_1^{(0)},u_2^{(0)}$ and a path $\gamma^{[v_1^{(0)},v_2^{(0)}]}$ and define $\mathbb{E}[\mu_{v_1^{(0)}}\mu_{v_2^{(0)}}\sigma_{u_1^{(0)}}\sigma_{u_2^{(0)}}]$ using the first way (i.e. use the notion of the random variable $\mu_{v_1^{(0)}}\mu_{v_2^{(0)}}$). Then, extend $\mathbb{E}[\mu_{v_1^{(0)}}\mu_{v_2^{(0)}}\sigma_{u_1^{(0)}}\sigma_{u_2^{(0)}}]$ and the path $\gamma^{[v_1,v_2]}$ to other vertices and faces step by step by the following procedure: if we defined $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ on v_1, v_2, u_1, u_2 and want to define it on v_1', v_2, u_1, u_2 where v_1 and v_1' are connected by an edge e then we extend the path $\gamma^{[v_1,v_2]}$ by the edge e (or subtract e from the path if it was presented there). In the same way we can replace v_2 by any neighbor. If we want to replace u_1 with its neighbor u_1' then we first check if the edge e between these two faces belongs to $\gamma^{[v_1,v_2]}$. If $e \not\subset \gamma^{[v_1,v_2]}$ then we just compute $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1'}\sigma_{u_2}]$ using the first definition above. Otherwise we compute $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ and multiply it by -1. Note that on the language of double covers the first definition of $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ says rouphly "take both spins from the first sheet and compute the correlation". When u_1 crosses the cut $\gamma^{[v_1,v_2]}$ then it jumps to the second sheet of the cover, so we have to switch the sign since all states are antisymmetric with respect to the double cover involution.

As defined above $\mathbb{E}[\mu_{v_1}\mu_{v_2}\sigma_{u_1}\sigma_{u_2}]$ becomes a branching function under this definition with the same branching structure as $\big[\prod_{p=1}^2\prod_{q=1}^2(v_p-u_q)\big]^{1/2}$.

Due to some technical reason (see Problem 4) it is convenient to study an observable defined on *corners* c, d by $\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}]$. If we define this function using the procedure above then it will have branching each time the corner c or the corner d rotates by 360°. To get rid of this local branching we multiply $\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}]$ by $\eta_c\eta_d$ where $\eta_c = e^{\frac{\pi i}{4}}\exp(-\frac{i}{2}\arg(v(c)-u(c)))$. Since arg is multiply defined the function η_c is multivalued but the product $\eta_c\eta_d\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}] = \mathbb{E}[\psi_c\psi_d]$ then can be defined as a single-valued function. Now let us show that $\mathbb{E}[\psi_c\psi_d] = -\mathbb{E}[\psi_d\psi_c]$. To show this we start moving c and d step by step such that eventually they interchange their position. Let us choose their trajectories to be disjoint from each other, and let us assume that the function $\arg(v(c)-u(c)) - \arg(v(d)-u(d))$ changed its value by $2\pi k$ in the end of this procedure. Then it is easy to observe that in the end of the day $\mathbb{E}[\mu_{v(c)}\mu_{v(d)}\sigma_{u(c)}\sigma_{u(d)}]$ was muliplied by $(-1)^{k+1}$. The anticommutativity follows. The case of $\mathbb{E}[\psi_c\psi_d\mathcal{O}_\varpi^{[\mu,\sigma]}]$ can be treated exactly in the same way.

Problem 4 (propagation equation for fermions). (a) Prove the propagation equation $X_\varpi(c_2) = X_\varpi(c_1)\cdot\cos\theta_e + X_\varpi(c_3)\cdot\sin\theta_e$ for Kadanoff-Ceva fermions.

(*Hint:* note that $\exp[-2\beta J_e\sigma_{u^\flat(e)}\sigma_{u^\sharp(e)}]\cdot\cos\theta_e + \sigma_{u^\flat(e)}\sigma_{u^\sharp(e)}\cdot\sin\theta_e = 1$.)

(b) Prove Smirnov's reformulation of the propagation equation for the critical model on isoradial graphs: provided that $z_e = (v^-(e)u^\flat(e)v^+(e)u^\sharp(e))$ is a *rhombus* with the half-angle θ_e and the Ising weights are chosen so that $x_e = \tan\frac{1}{2}\theta_e$, the propagation equation on this rhombus is equivalent to the existence of a value $\Psi_\varpi(z_e)\in\mathbb{C}$ such that

$$\Psi_\varpi(c) \;=\; \tfrac{1}{2}[\Psi_\varpi(z) + \eta_c^2\cdot\overline{\Psi_\varpi(z)}] = \mathrm{Proj}[\Psi_\varpi(z);\eta_c\mathbb{R}].$$

Solution. (a) Recall that $X_\varphi(c) = \mathbb{E}[\mu_{v(c)}\sigma_{u(c)}\mathcal{O}_\varpi^{[\mu,\sigma]}]$ is defined via the procedure described in the solution to Problem 3. In particular, for three consequetive corners c_1, c_2, c_3 we know the particular way (based on the choice of paths γ^v, γ_u) to interprete $\mu_{v(c)}\sigma_{u(c)}\mathcal{O}_\varpi^{[\mu,\sigma]}$ as a random variable. The relation in the Hint immediately implies that

$$\mu_{v(c_2)}\sigma_{u(c_2)}\mathcal{O}_\varpi^{[\mu,\sigma]} = \mu_{v(c_1)}\sigma_{u(c_1)}\mathcal{O}_\varpi^{[\mu,\sigma]}\cos\theta_e + \mu_{v(c_3)}\sigma_{u(c_3)}\mathcal{O}_\varpi^{[\mu,\sigma]}\sin\theta_e$$

from which we conclude the propagation equation.

(b) It is clear that the existance of such a function Ψ_ϖ on rombus implies the propagation equation. Vise versa, using the procedure from the solution to the problem 3 we define a complex-valued function Ψ_ϖ on rombus by

$$\Psi_\varpi(z_e) = \mathbb{E}[\psi_{(v^-(e)u^\flat(e))}] + \mathbb{E}[\psi_{(v^+(e)u^\sharp(e))}].$$

It is clear from the definition that if $c = (v^-(e)u^\flat(e))$ or $c = (v^+(e)u^\sharp(e))$ then $\Psi_\varpi(c) = \mathrm{Proj}[\Psi_\varpi(z); \eta_c\mathbb{R}]$. On the other hand, the propagation equation implies that

$$\mathbb{E}[\psi_{(v^-(e)u^\flat(e))}] + \mathbb{E}[\psi_{(v^+(e)u^\sharp(e))}] = \mathbb{E}[\psi_{(u^\sharp(e)v^-(e))}] + \mathbb{E}[\psi_{(u^\flat(e))v^+(e))}]$$

so that the desired relation holds for the other two corners too.