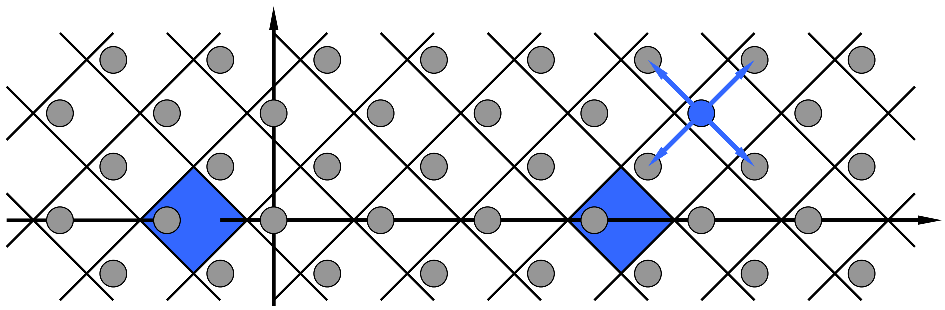
The goal of this problem set is to compute the limit of the *infinite-volume* 'diagonal' two-point functions $D_{n+1}(x) := \mathbb{E}_{\mathbb{C}^\diamond}[\sigma_{(-\frac{3}{2},0)}\sigma_{(2n+\frac{1}{2},0)}]$ for $x = \tan\frac{1}{2}\theta$, $\theta < \frac{\pi}{4}$:

$$D_{n+1} \;\to\; (1-\tan^4\theta)^{1/4} \quad \text{as} \quad n\to\infty.$$

(this is a version of the famous Onsager–Kaufman–Yang theorem).



We also use the notation $D_n := D_n(x)$, $D_n^* := D_n(x^*)$, where $x^* := \tan\frac{1}{2}(\frac{\pi}{4}-\theta)$. Similarly to the critical point $x_{\mathrm{crit}} = \tan\frac{\pi}{8} = \sqrt{2}-1$, we work with the observable

$$V(k,s) := \langle \chi_{(k,s)}\mu_{(-\frac{1}{2},0)}\sigma_{(2n+\frac{1}{2},0)}\rangle, \quad k,s\in\mathbb{Z},\ k+s\in 2\mathbb{Z}. \tag{1}$$

Recall that V satisfies the *massive harmonicity* condition (with $m := \sin 2\theta < 1$):

$$\Delta^{(m)}V(k,s) := \frac{m}{4}\sum_{\pm,\pm} V(k\pm1, s\pm1) - V(k,s) = 0, \quad (k,s)\neq(0,0),(2n,0).$$

- additional details on how to pass from the three-term propagation equation to the massive harmonicity can be found in [Section 2.4, `arXiv:1904.09168`];
- the computation of D_n at the critical temperature (Wu's formula) can be found in the Appendix of the same paper, see also [Section 3, `arXiv:1605.0903`];

Problem 1. Prove that, for $s\geq 0$,

$$V(k,s) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\frac{k}{2}t}(y(t))^s Q_n(e^{it})dt, \qquad y(t) = \frac{1-(1-m^2\cos^2(\frac{1}{2}t))^{1/2}}{m\cos(\frac{1}{2}t)},$$

where $Q_n(z) = D_n + \ldots + D_n^* z^n$ is a polynomial of degree n with prescribed leading and free terms and such that it is orthogonal to $z,\ldots,z^{n-1}$ with respect to the weight

$$w(e^{it}) := (1+q^2)\cdot(1-m^2\cos^2(\tfrac{1}{2}t))^{1/2}, \quad q := \tan\theta < 1,$$

on the unit circle $z = e^{it}$ (note that these properties define Q_n uniquely).

Solution. Note first of all that $V(s,t)$ is not properly defined by (1) because the spin-disorder correlation $\langle \chi_{(k,s)}\mu_{(-\frac{1}{2},0)}\sigma_{(2n+\frac{1}{2},0)}\rangle$ is a priory defined only up to a sign. Let us choose the sign of $V(0,0)$ such that $V(0,0) = D_n$. Then we can extend V to all other corners of the same type moving the corner (s,t) step by step in such a way that it never make a full turn; as a result we obtain a function that branches only around two blue faces (see the picture). Let us perform a cut along the corners $(0,0),(0,2),\ldots,(0,2n)$ and consider V as a single-valued function outside this cut. One can check that the obtain function satisfies $V(k,-s) = -V(k,s)$, therefore we conclude that $V(k,0) = 0$ if $k < 0$ or $k > 2n$. Now, let us instead consider the single-valued extension of V outside of the union of two cuts L_-, L_+ made along corners $\dots, (-2,0), (0,0)$ and $(2n,0), (2n+2,0), \dots$ respectively — in this case we have $V(k,-s) = V(k,s)$. This extension satisfies the following properties

- V is bounded (since it is an expectation of a bounded variable),
- V is massive harmonic at each corner outside $L_- \cup L_+$,
- $V(0,0) = D_n$, $V(2n,0) = D_n^*$ and $V(s,t) = 0$ for other corners from $L_- \cup L_+$.

We claim that V is uniquely defined by these three properties. Indeed, assume that W is another function satisfying these properties, consider the function $F = V - W$, then F is identically zero along the boundary and is massive harmonic outside cuts:

$$F(k,s) = \frac{m}{4} \sum_{\pm,\pm} F(k\pm1, s\pm1).$$

Repeating this for each $(k\pm1, s\pm1)$ if it does not lie on $L_- \cup L_+$ and using the boundedness of F and the fact that simple random walk is recurrent in 2D we find that

$$F(k,s) = \mathbb{E}[m^\tau F(X_\tau)] = 0$$

where X is the simple random walk started at (s,t) and τ is the stopping time indicating the first time when X hits $L_- \cup L_+$.

Now, consider the function W defined by $W(k,s) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\frac{k}{2}t}(y(t))^s Q_n(e^{it})dt$ if $s \geq 0$ and by $W(k,s) = W(k,-s)$ if $s < 0$. Given $|s| > 0$ then one can check that

$$\Delta^{(m)}W(k,s) =$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \Big[\frac{m}{2}\cos(t/2)\big(y(t) + y(t)^{-1}\big) - 1\Big] e^{-i\frac{k}{2}t}(y(t))^s Q_n(e^{it})dt = 0$$

since y satisfies $\frac{m}{2}\cos(t/2)\big(y(t) + y(t)^{-1}\big) = 1$. Similarly, one can check that

$$\Delta^{(m)}W(k,0) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\frac{k}{2}t} Q_n(e^{it})\, w(e^{it})dt.$$

Using that Q_n is orthogonal to $z, \dots, z^{n-1}$ we get that $\Delta^{(m)}W(k,0) = 0$ for $0 < k < 2n$ and thus for all corners outside $L_- \cup L_+$.

Finally, using that Q_n is degree n polynomial we see that W satisfies the same boundary conditions as V. Thus $V = W$.

Problem 2 (this is an unpleasant local computation). (a) For $n \geq 1$, prove that

$$w(e^{it})Q_n(e^{it}) = \dots + D_{n+1} + 0 + q^2 D_{n+1}^* e^{int} + \dots \tag{2}$$

(b) For $n = 0$, argue that the constant term in the Fourier series of $w(e^{it})Q_0(e^{it})$ is $D_1 + q^2 D_1^*$.

Solution. This a solution was pleasantly written by the members of group C8, thanks a lot!

If we recall from problem (1) that $\Delta^{(m)}V(k,0) = -\frac{1}{1+q^2}\frac{1}{2\pi}\int_0^{2\pi} w(e^{it})Q_n(e^{it})e^{-i\frac{k}{2}t}dt$. Then the fact that the laplacian of $V(k,0)$ is zero when $k = 1, \dots, n-1$ gives the desired zeros in the Fourier series of $w(e^{it})Q_n(e^{it})$. Now we need to find the constant term and the $k = n$ term.

We label our corners as follows: c is the corner at which we are taking the laplacian, $c_+^\sharp$ is up-right, $c_+^\flat$ is bottom-right, $c_-^\sharp$ is top-left, and $c_-^\flat$ is bottom-left (all these corners are to the right of their vertex). Also, let a_1 be the corner sharing a vertex with c but left of the vertex, a_2 be the corner sharing a vertex with $c_+^\sharp$ but left of the vertex, a_3 be the corner sharing a face with c but on the right, and a_4 be the corner sharing a vertex with $c_+^\flat$ but left of the vertex.

Suppose we are away from $(0,0)$ an $(2n,0)$. Then repeated applications of the propagation equation (using some auxiliary corners not labelled) we end up with a system of equations

$$X_\omega(c_-^\sharp)\sin(\theta) - X_\omega(c)\cos(\theta) = -X_\omega(a_2)\sin(\theta) + X_\omega(a_1)\cos(\theta) \tag{3}$$

$$X_\omega(c)\sin(\theta) - X_\omega(c_+^\sharp)\cos(\theta) = X_\omega(a_3)\sin(\theta) - X_\omega(a_2)\cos(\theta) \tag{4}$$

$$X_\omega(c)\sin(\theta) - X_\omega(c_+^\flat)\cos(\theta) = -X_\omega(a_3)\sin(\theta) + X_\omega(a_4)\cos(\theta) \tag{5}$$

$$X_\omega(c_-^\flat)\sin(\theta) - X_\omega(c)\cos(\theta) = X_\omega(a_4)\sin(\theta) - X_\omega(a_1)\cos(\theta) \tag{6}$$

where $\omega = \{\mu_{(-1/2,0)}, \sigma_{(2n+1/2,0)}\}$. Taking the linear combination $(1)\times\cos(\theta) - (2)\times\sin(\theta) - (3)\times\sin(\theta) + (4)\times\cos(\theta)$ all of the RHS cancels and after some simplifying we are left with

$$\frac{\sin(2\theta)}{2}\left(X_\omega(c_-^\sharp) + X_\omega(c_+^\sharp) + X_\omega(c_-^\flat) + X_\omega(c_+^\flat)\right) - 2X_\omega(c) = 0.$$

We see that $\Delta^{(m)}X_\omega(c) = 0$ when c is away from $(0,0)$ and $(2n,0)$.

Now instead, let's suppose c is at $(0,0)$. Now when we take a loop around the vertex with disorder $\mu_{(-1/2,0)}$, we will get an opposite sign from above. In particular, $X_\omega(a_1) \mapsto -X_\omega(a_1)$ in equation (4). Taking the same sum of the equations as before, we get

$$\frac{\sin(2\theta)}{2}\left(X_\omega(c_-^\sharp) + X_\omega(c_+^\sharp) + X_\omega(c_-^\flat) + X_\omega(c_+^\flat)\right) - 2X_\omega(c) = 2X_\omega(a_1)\cos^2(\theta).$$

Note that $X_\omega(a_1) = \mathbb{E}[\chi_{a_1}\mu_{(-1/2,0)}\sigma_{(2n+1/2,0)}] = \mathbb{E}[\mu_{(-1/2,0)}\mu_{(-1/2,0)}\sigma_{(-3/2,0)}\sigma_{(2n+1/2,0)}] = D_{n+1}$. Using how the laplacian of V is related to $w(e^{it})Q_n(e^{it})$ we get

$$\frac{1}{2\pi}\int_0^{2\pi} w(e^{it})Q_n(e^{it})dt = -D_{n+1}.$$

Now to if $k = 2n$, we have to change signs when going around the face at $(2n + \frac{1}{2}, 0)$. This changes the sign of $X_\omega(a_3)$ in eqn (3). Taking the same linear combination of our equations we're left with

$$\frac{\sin(2\theta)}{2}\left(X_\omega(c_-^\sharp) + X_\omega(c_+^\sharp) + X_\omega(c_-^\flat) + X_\omega(c_+^\flat)\right) - 2X_\omega(c) = -2X_\omega(a_3)\sin^2(\theta).$$

Note $X_\omega(a_3) = \mathbb{E}[\mu_{(-1/2,0)}\mu(2n+3/2,0)\sigma_{(2n+1/2,0)}\sigma_{(2n+1/2,0)}] = \mathbb{E}^*[\sigma^*_{(-1/2,0)}\sigma^*_{(2n+3/2,0)}] = D^*_{n+1}$. Just as before this gives

$$\frac{1}{2\pi}\int_0^{2\pi} w(e^{it})Q_n(e^{it})e^{-int}dt = tan^2(t)D^*_{n+1}$$

Lastly, if $n = 0$ and we look at $k = 0$ we need to change the sign of $X_\omega(a_1) \mapsto -X_\omega(a_1)$ in eqn (4) and $X_\omega(a_3)$ in eqn (3) to deal with the vertex at $(-\frac{1}{2}, 0)$ and the face at $(\frac{1}{2}, 0)$. This will result in both extra terms from the above calculations. So we'll get

$$\frac{1}{2\pi}\int_0^{2\pi} w(e^{it})Q_0(e^{it})dt = -D_1 + tan^2(t)D_1^*.$$

Let $\Phi_n(z) = z^n + \ldots = \overline{\Phi_n(\overline{z})}$ be the n-th orthogonal polynomial with respect to $w(e^{it})$. Recall the recurrence relation $\Phi_{n+1}(z) = z\Phi_n(z) - \alpha_n\Phi_n^*(z)$, where $\Phi_n^*(z) = z^n\Phi_n(z^{-1})$, and $\alpha_n = \overline{\alpha}_n$ are *Verblunski coefficients*, see Section 2 in the reference quoted above. Recall also that $\beta_n := \|\Phi_n\|^2 = \|\Phi_n^*\|^2 = \beta_0\prod_{k=0}^{n-1}(1-\alpha_k^2)$, where the norms are taken wrt $\frac{1}{2\pi}w(e^{it})dt$.

Problem 3. **(a)** Prove the recurrence relation

$$\begin{pmatrix} D_{n+1} \\ q^2D_{n+1}^* \end{pmatrix} = \beta_{n-1}\begin{pmatrix} 1 & \alpha_{n-1} \\ \alpha_{n-1} & 1 \end{pmatrix}\begin{pmatrix} D_n \\ D_n^* \end{pmatrix}, \quad n \geq 1.$$

(b) By induction deduce the identity $D_{n+1}\Phi_n^*(q^2) + q^2D_{n+1}^*\Phi_n(q^2) = \beta_n \ldots \beta_0$.

Solution. Substituting $z = 0$ we get $\alpha_n = -\Phi_{n+1}(0)$. Note that Φ_n, Φ_n^* span the space of degree n polynomials that are orthogonal to $z, \ldots, z^{n-1}$ with respect to the weight w. It follows that

$$Q_n = c_n\Phi_n + c_n^*\Phi_n^*.$$

and c_n, c_n^* satisfy

$$\begin{pmatrix} D_n \\ D_n^* \end{pmatrix} = \begin{pmatrix} 1 & -\alpha_{n-1} \\ -\alpha_{n-1} & 1 \end{pmatrix}\begin{pmatrix} c_n^* \\ c_n \end{pmatrix}$$

Notice that $\|\Phi_n\|^2 = \langle\Phi_n, z^n\rangle = \langle\Phi_n^*, 1\rangle = \|\Phi_n^*\|^2$, where all scalar products are taken in $L^2(w(e^{it})dt)$. Using this observation, the fact that $c_n = \langle Q_n, z^n\rangle$ and $c_n^* = \langle Q_n, 1\rangle$ and (2) we get that

$$\beta_n\begin{pmatrix} c_n^* \\ c_n \end{pmatrix} = \begin{pmatrix} D_{n+1} \\ q^2D_{n+1}^* \end{pmatrix}$$

Composing these we get the desired relation.

We now take for granted that $D_n^* = D_n^*(x^*) \leq D_n(x_{\text{crit}}) \to 0$ as $n \to \infty$.

Problem 4. Check that $w(e^{it}) = |1 - q^2e^{it}|$. Prove that

$$D_{n+1} \;\to\; \frac{\prod_{k=0}^{\infty}\beta_k}{\lim_{n\to\infty}\Phi_n^*(q^2)} \;=\; \frac{(1-q^4)^{-1/4}}{(1-q^4)^{-1/2}} \;=\; (1-q^4)^{1/4} \quad \text{as} \quad n \to \infty$$

due to the Szegö theorems (see Section 8 in the reference quoted above).

Solution. Using the Szegö recurrence relation and the recurrence relation obtained in Problem 3 one can check that

$$\begin{aligned} q^2D_{n+1}\Phi_{n+1}(q^2) + D_{n+1}\Phi_n^*(q^2) &= \beta_n \cdot (q^2D_n^*\Phi_{n-1}(q^2) + D_n\Phi_{n-1}^*(q^2)) \\ &= \beta_n \ldots \beta_1 \cdot (q^2D_1^* + D_1) = \beta_n \ldots \beta_0 \end{aligned}$$

where in the last step we used the result of Problem 2(b). It follows that

$$D_{n+1} = \frac{\beta_n \ldots \beta_0}{\Phi_n^*(q^2)} - \frac{\Phi_n(q^2)}{\Phi_n^*(q^2)}D_{n+1}^*.$$

The first and the second Szegö theorems (see the supplementary material) imply that

$$\lim_{n\to+\infty} \frac{\beta_n \dots \beta_0}{\Phi_n^*(q^2)} = \frac{(1-q^4)^{-1/4}}{(1-q^4)^{-1/2}} \;=\; (1-q^4)^{1/4}.$$

It remains to show that $\lim\limits_{n\to+\infty} \frac{\Phi_n(q^2)}{\Phi_n^*(q^2)} D_{n+1}^* = 0$. We know that $D_{n+1}^* \to 0$, thus it is enough to show that $\frac{\Phi_n(q^2)}{\Phi_n^*(q^2)}$ is bounded. The first Szegö theorem ensures that $\Phi_n^*(q^2)$ is bounded from below. To see that $\Phi_n(q^2)$ is bounded from above let us write

$$\left|\Phi_n(q^2)\right| = \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{\Phi_n(e^{it})}{e^{it}-q^2}\, e^{it} dt\right| \le \beta_n^{1/2} \cdot \frac{1}{2\pi}\left(\int_0^{2\pi} \frac{dt}{w(e^{it})|e^{it}-q^2|^2}\right)^{1/2}.$$

Using that w is bounded from below and β_n has a finite limit (due to the first Szegö theorem) we get the boundedness.

For a nice proof of the strong Szegö theorem (the value $\prod_{k=0}^{\infty} \beta_k$) see
A Fredholm determinant formula for Toeplitz determinants
by Alexei Borodin and Andrei Okounkov, `arXiv:math/9907165`
