Let $\Omega \subset \mathbb{C}$ be a bounded (not necessarily simply connected) domain, $a \in \Omega$ and $|\eta| = 1$. Recall that $f^{[\eta]}(a, \cdot) : \Omega \setminus \{a\} \to \mathbb{C}$ is defined as the unique[!] *holomorphic* function such that

$$f^{[\eta]}(a, z) = \frac{\overline{\eta}}{z - a} + O(1) \quad \text{as} \quad z \to a, \qquad f^{[\eta]}(a, \zeta) \in (\tau(\zeta))^{-1/2}\mathbb{R}, \quad \zeta \in \partial\Omega,$$

where $\tau(\zeta)$ denotes the tangent vector to Ω (oriented so that Ω remains to the left of $\tau(\zeta)$).

Problem 1. (a) Prove that there exists (unique) functions $f(a, \cdot)$ and $f^\star(a, \cdot)$ such that

$$f^{[\eta]}(a, z) = \tfrac{1}{2}[\,\overline{\eta} f(a, z) + \eta f^\star(a, z)\,] \quad \text{for all} \quad z \in \Omega \text{ and } |\eta| = 1.$$

(b) Denote $f^{[\eta,\mu]}(w, z) := \mathrm{Re}[\overline{\mu} f^{[\eta]}(w, z)]$, where $w \neq z$ and $|\eta| = |\mu| = 1$. Prove that

$$f^{[\mu,\eta]}(z, w) = -f^{[\eta,\mu]}(w, z).$$

Hint: Consider $\oint_{\partial\Omega} f^{[\eta]}(w, \zeta) f^{[\mu]}(z, \zeta) d\zeta$.

(c) Deduce that $f(z, w) = -f(w, z)$ and $f^\star(z, w) = -\overline{f^\star(w, z)}$. In particular, $f(z, w)$ is holomorphic in both variables (except at $z = w$) whilst $f^\star(w, z)$ is holomorphic in z and anti-holomorphic in w. Argue that the definition

$$\langle \varepsilon_w \rangle_\Omega^+ \; := \; \tfrac{i}{2} f^\star(w, w)$$

makes sense and that $\langle \varepsilon_w \rangle_\Omega^+ \in \mathbb{R}$.

(d) Prove the conformal covariance rules: if $\varphi : \Omega \to \Omega'$ is a conformal map, then

$$f_\Omega(w, z) = f_{\Omega'}(\varphi(w), \varphi(z)) \cdot (\varphi'(w))^{1/2} (\varphi'(z))^{1/2},$$
$$f^\star_\Omega(w, z) = f^\star_{\Omega'}(\varphi(w), \varphi(z)) \cdot (\overline{\varphi'(w)})^{1/2} (\varphi'(z))^{1/2}.$$

In particular, one has $\langle \varepsilon_w \rangle_\Omega^+ = \langle \varepsilon_{\varphi(w)} \rangle_{\Omega'}^+ \cdot |\varphi'(w)|$.

Solution. (a) Obviously, if $f, f^\star$ exists, then they must satisfy

$$f^{[1]}(a, z) = \frac{1}{2}(\, f(a, z) + i f^\star(a, z)\,),$$
$$f^{[i]}(a, z) = -\frac{i}{2}(\, f(a, z) - f^\star(a, z)\,).$$

This system has a unique solution given by

$$f(a, z) = f^{[1]}(a, z) + i f^{[i]}(a, z),$$
$$f^\star(a, z) = f^{[1]}(a, z) - i f^{[i]}(a, z).$$

Using that $f^{[\eta]}$ is real linear with respect to η (proven in lectures) we find that

$$f^{[\eta]}(a, z) = f^{[1]}(a, z)\mathrm{Re}\eta + f^{[i]}(a, z)\mathrm{Im}\eta = \frac{1}{2}[\,\overline{\eta} f(a, z) + \eta f^\star(a, z)\,], \tag{1}$$

thus $f, f^\star$ indeed satisfy desired properties. Note that if $f, f^\star$ satisfy (1) for any η then we have $f(a, z) = \partial/\partial\overline{\eta} f^{[\eta]}(a, z)$ and $f^\star(a, z) = \frac{\partial}{\partial\eta} f^{[\eta]}(a, z)$.

(b) Observe that

$$\oint_{\partial\Omega} f^{[\eta]}(w,\zeta) f^{[\mu]}(z,\zeta) d\zeta = 2\pi i (\overline{\eta} f(z,w) + \overline{\mu} f(w,z)).$$

Using that $\mathrm{Im} \oint_{\partial\Omega} f^{[\eta]}(w,\zeta) f^{[\mu]}(z,\zeta) d\zeta = 0$ due to the boundary conditions we get the claim.

(c) Note that $f(z,w) = \frac{\partial}{\partial\overline{\mu}} \frac{\partial}{\partial\overline{\eta}} f^{[\eta,\mu]}(z,w)$, whereas $f^\star(z,w) = \frac{\partial}{\partial\overline{\mu}} \frac{\partial}{\partial\eta} f^{[\eta,\mu]}(z,w)$ and $\overline{f^\star(z,w)} = \frac{\partial}{\partial\mu} \frac{\partial}{\partial\overline{\eta}} f^{[\eta,\mu]}(z,w)$. Using these observations and (b) we immediately get the result.

(d) Consider the function $f_{\Omega'}^{[\eta\cdot(\overline{\phi'(w)})^{1/2}]}(\phi(w),\phi(w)) \cdot (\phi'(z))^{1/2}$. It satisfies the same properties as $f_\Omega^{[\eta]}(w,z)$, thus we have $f_{\Omega'}^{[\eta\cdot(\overline{\phi'(w)})^{1/2}]}(\phi(w),\phi(w)) \cdot (\phi'(z))^{1/2} = f_\Omega^{[\eta]}(w,z)$ due to the uniqueness of $f_\Omega^{[\eta]}(w,z)$ and we can write

$$\begin{aligned}
&\tfrac{1}{2}[\,\overline{\eta} f_\Omega(w,z) + \eta f_\Omega^\star(w,z)\,] = f_\Omega^{[\eta]}(w,z) = \\
&= f_{\Omega'}^{[\eta\cdot(\overline{\phi'(w)})^{1/2}]}(\phi(w),\phi(w)) \cdot (\phi'(z))^{1/2} = \\
&= \tfrac{1}{2}[\,\overline{\eta} f_{\Omega'}(\varphi(w),\varphi(z)) \cdot (\varphi'(w))^{1/2}(\varphi'(z))^{1/2} + \eta f_{\Omega'}^\star(\varphi(w),\varphi(z)) \cdot (\overline{\varphi'(w)})^{1/2}(\varphi'(z))^{1/2}\,].
\end{aligned}$$

Using that $f(a,z)$ and $f^\star(a,z)$ are uniquely defined we get the result

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Recall that the *holomorphic spinor* $g_{[v,u]}(z)$ (defined on the double cover of Ω ramified over $u,v \in \Omega$, $u \neq v$) is uniquely characterized by the following conditions:

$$g_{[v,u]}(z) \;=\; \frac{e^{-i\frac{\pi}{4}}}{\sqrt{z-v}} \cdot [\,1 + O(z-v)\,] \quad \text{as} \quad z \to v; \tag{2}$$

$$g_{[v,u]}(z) \;=\; \frac{e^{i\frac{\pi}{4}}}{\sqrt{z-u}} \cdot [\,c + O(z-u)\,] \quad \text{as} \quad z \to u, \quad \text{with an unknown} \quad c \in \mathbb{R}, \tag{3}$$

and the boundary conditions $g_{[v,u]}(\zeta) \in (\tau(\zeta))^{-1/2}\mathbb{R}$ for $\zeta \in \partial\Omega$. Further, recall that $\mathcal{A}(v,u)$ is defined as the next coefficient in the expansion of $g_{[v,u]}(z)$ as $z \to v$:

$$g_{[v,u]}(z) \;=\; \frac{e^{-i\frac{\pi}{4}}}{\sqrt{z-v}} \cdot [\,1 + 2\mathcal{A}(v,u)(z-v) + O((z-v)^2)\,],$$

and that

$$\langle \sigma_u \sigma_v \rangle_\Omega^+ \;:=\; \exp\left[\int \mathrm{Re}\big[\mathcal{A}(v,u)dv + \mathcal{A}(u,v)du\big]\right],$$

where the multiplicative normalization is chosen so that $\langle \sigma_u \sigma_v \rangle_\Omega^+ \sim |u-v|^{-1/4}$ as $u \to v$.

Problem 2. The goal is to prove the *fusion rule* $\sigma\sigma \rightsquigarrow 1 + \frac{1}{2}\varepsilon + \ldots$, more precisely:

$$\langle \sigma_v \sigma_u \rangle_\Omega^+ \;=\; |v-u|^{-1/4} \cdot [1 + \tfrac{1}{2}\langle \varepsilon_v \rangle_\Omega^+ \cdot |v-u| + o(|v-u|)] \quad \text{as} \quad v \to u \tag{4}$$

(not using explicit expressions available in simply connected Ω), where the correlation functions $\langle \sigma_u \sigma_v \rangle_\Omega^+$ and $\langle \varepsilon \rangle_\Omega^+$ are defined above.

Denote $\eta := e^{i\frac{\pi}{4}} \cdot (\overline{v} - \overline{u})^{1/2}/|v-u|^{1/2}$. First, take for granted that $\langle \varepsilon_v \rangle_\Omega^+ \to \langle \varepsilon_u \rangle_\Omega^+$ and

$$g_{[v,u]}(z) \;=\; |v-u|^{1/2} \cdot \left[f^{[\eta]}(v,z) \cdot \left(\frac{z-v}{z-u}\right)^{1/2} + \; o(1) \right] \quad \text{as} \quad v \to u, \tag{5}$$

uniformly on compact subsets $z \in \Omega \setminus \{u\}$.

Remark: The right-hand side of (5) is chosen so that the difference does *not* blow up at $z = v$ and approximately satisfies (3) and the boundary conditions, so it should be small.

(a) Deduce from (5) that

$$2\mathcal{A}_{[v,u]} + \frac{1}{2(v-u)} \;=\; \langle \varepsilon_v \rangle_\Omega^+ \cdot \frac{|v-u|}{v-u} + o(1) \quad \text{as} \quad v \to u.$$

Hint: Consider $\oint g_{[v,u]}(z) \cdot (z-u)^{1/2}(z-v)^{-3/2} dz$.

(b) Deduce (4) from (5) and the asymptotics $\langle \sigma_u \sigma_v \rangle_\Omega^+ \sim |v-u|^{-1/4}$ as $v \to u$.

(c) Prove that $\langle \varepsilon_v \rangle_\Omega^+ \to \langle \varepsilon_u \rangle_\Omega^+$ as $v \to u$.

Hint: Argue that each subsequential limit of $f^{[\eta]}(v, \cdot)$ must coincide with $f^{[\eta]}(u, \cdot)$.

(d)* Prove (5).

Solution. We begin with the following observation. Let $\overline{\Omega}$ be the domain in $\mathbb{C}$ obtained by reflecting Ω with respect to the real axis. Consider the functions $\tilde{f} : \overline{\Omega} \times \Omega \to \mathbb{C}$ defined by $\tilde{f}(v, z) = f^\star(\overline{v}, z)$. Notice that $f^{[1]}(w, z) - i f^{[i]}(w, z)$ has a removable singularity at $z = w$, hence $\tilde{f}$ is defined on the whole $\overline{\Omega} \times \Omega$. Due to the result of Problem 1(c) the function $\tilde{f}$ is "separately" holomorphic, i.e. it is holomorphic is each variable when the other one is fixed. It follows from the Hartogs theorem that $\tilde{f}$ is holomorphic as a function of two variables, hence we coclude that $f^\star(v, z)$ is analytic in $\overline{v}, z$. In the same way we find that $f(v, z) - \frac{2}{z-v}$ is holomorphic function in two variables. Notice that $f(v, z) - \frac{2}{z-v}$ is antisymmetric, hence vanishes on the diagonal.

(a) Using the properties of $f^{[\eta]}$ and the fact that $f(v, z) = -f(z, v)$ we find that

$$f^{[\eta]}(v, z) = \frac{e^{-i\frac{\pi}{4}}(v-u)^{1/2}}{|v-u|^{1/2}(z-v)} + e^{i\frac{\pi}{4}} \cdot (\overline{v} - \overline{u})^{1/2} / |v-u|^{1/2} \langle \varepsilon_v \rangle_\Omega^+ + O(z-v)$$

as $z \to v$. Substituting this relation and

$$\left(\frac{z-v}{z-u}\right)^{1/2} = \frac{(z-v)^{1/2}}{(v-u)^{1/2}} \left(1 - \frac{z-v}{2(v-u)} + O(z-v)^2\right)$$

into (5) we get the desired asymptotics.

(b) Due to the previous exercise we have

$$\int \mathrm{Re}\big[\mathcal{A}(v,u)dv + \mathcal{A}(u,v)du\big] = \int \mathrm{Re}\Big[-\frac{d(v-u)}{4(v-u)} + O(1)\Big].$$

The claim follows.

(c) As we mentioned above, the function $f^\star(v, z)$ is analytic in variables $\overline{v}, z$ and therefore it is continuous. It follows that $\langle \varepsilon_v \rangle_\Omega^+ = f^\star(v, v)$ is continuous too.

(d) Assume that u is fixed and $u-v$ is small and consider the function

$$F(z;v,u) := \left(g_{[v,u]}(z) \ - \ |v-u|^{1/2} \cdot f^{[\eta]}(v,z) \cdot \left(\frac{z-v}{z-u}\right)^{1/2} \right)^2 .$$

When z belongs to the boundary of Ω we have

$$\left(\frac{z-v}{z-u}\right)^{1/2} = 1 + O(v-u).$$

Using the boundary conditions of $g_{[v,u]}$ and $f^{[\eta]}(v,z)$ we find that

$$\int_{\partial\Omega} F(z;v,u)\,dz = \int_{\partial\Omega} |F(z;v,u)|\,|dz| + O(v-u)$$

On the other hand, we have

$$\int_{\partial\Omega} F(z;v,u)\,dz = 2\pi i \left(e^{i\frac{\pi}{4}}c - |v-u|^{1/2} f^{[\eta]}(v,u)\cdot(u-v)^{1/2}\right)^2 .$$

Using that

$$f^{[\eta]}(v,u) = \frac{e^{i\frac{\pi}{4}}(u-v)^{1/2}}{|v-u|^{1/2}(u-v)}\,(1+O(v-u))$$

we find that

$$\int_{\partial\Omega} F(z;v,u)\,dz = -2\pi(c-1)^2 + O(v-u).$$

Comparing these two expressions for the integral we get

$$\int_{\partial\Omega} |F(z;v,u)|\,|dz| + 2\pi(c-1)^2 = O(v-u).$$

Now, let $\Omega_{v,u} = \Omega \setminus \{z \ : \ |z-u| \le |v-u|^2\}$. It follows that there exists a constant $C>0$ such that

$$\int_{\partial\Omega_{v,u}} |F(z;v,u)|\,|dz| \le C|v-u|.$$

Since F if holomorphic in $\Omega_{v,u}$ we conclude that for any compact $K \subset \Omega \setminus \{u\}$ there exists a constant C' that such that

$$\max_{z\in K} |F(z;v,u)| \le C'|v-u|$$

for provided $|v-u|$ is small enough and (5) follows.

More information on correlations of $\psi, \mu, \sigma, \varepsilon$ and fusion rules: [Section 4, `arXiv:1605.09035`]