The hitchhiker’s guide to the (critical) planar Ising model. TA3.

Let \( \Omega \subset \mathbb{C} \) be a bounded (not necessarily simply connected) domain, \( a \in \Omega \) and \( |\eta| = 1 \). Recall that \( f^{[\eta]}(a, \cdot) : \Omega \setminus \{a\} \to \mathbb{C} \) is defined as the unique\(!\) holomorphic function such that
\[
f^{[\eta]}(a, z) = \frac{\eta f(a, \zeta)}{\overline{z} - a} + O(1) \quad \text{as} \quad z \to a, \quad f^{[\eta]}(a, \zeta) \in (\tau(\zeta))^{-1/2} \mathbb{R}, \quad \zeta \in \partial \Omega,
\]
where \( \tau(\zeta) \) denotes the tangent vector to \( \Omega \) (oriented so that \( \Omega \) remains to the left of \( \tau(\zeta) \)).

**Problem 1.** (a) Prove that there exists (unique) functions \( f(a, \cdot) \) and \( f^*(a, \cdot) \) such that
\[
f^{[\eta]}(a, z) = \frac{1}{2} [ \eta f(a, z) + \eta f^*(a, z) ] \quad \text{for all} \quad z \in \Omega \quad \text{and} \quad |\eta| = 1.
\]
(b) Denote \( f^{[\eta,\mu]}(w, z) := \text{Re}[\overline{\mu} f^{[\eta]}(w, z)] \), where \( w \neq z \) and \( |\eta| = |\mu| = 1 \). Prove that
\[
f^{[\eta,\mu]}(z, w) = -f^{[\eta,\mu]}(w, z).
\]
Hint: Consider \( f_{\partial \Omega} f^{[\eta]}(w, \zeta) f^{[\eta]}(z, \zeta) d\zeta \).

(c) Deduce that \( f(z, w) = -f(w, z) \) and \( f^*(z, w) = -f^*(w, z) \). In particular, \( f(z, w) \) is holomorphic in both variables (except at \( z = w \)) whilst \( f^*(w, z) \) is holomorphic in \( z \) and anti-holomorphic in \( w \). Argue that the definition
\[
\langle \varepsilon_w \rangle_{\Omega}^+ := \frac{i}{2} f^*(w, w)
\]
makes sense and that \( \langle \varepsilon_w \rangle_{\Omega}^+ \in \mathbb{R} \).

(d) Prove the conformal covariance rules: if \( \varphi : \Omega \to \Omega' \) is a conformal map, then
\[
\begin{align*}
f_{\Omega}(w, z) &= f_{\Omega'}(\varphi(w), \varphi(z)) \cdot (\varphi'(w))^{1/2}(\varphi'(z))^{1/2}, \\
f^*_{\Omega}(w, z) &= f^*_{\Omega'}(\varphi(w), \varphi(z)) \cdot (\overline{\varphi'(w)})^{1/2}(\overline{\varphi'(z)})^{1/2}.
\end{align*}
\]
In particular, one has \( \langle \varepsilon_w \rangle_{\Omega}^+ = \langle \varepsilon_{\varphi(w)} \rangle_{\Omega'}^+ \cdot |\varphi'(w)| \).

**Solution.** (a) Obviously, if \( f, f^* \) exists, then they must satisfy
\[
\begin{align*}
f^{[1]}(a, z) &= \frac{1}{2} ( f(a, z) + if^*(a, z) ), \\
f^{[i]}(a, z) &= -\frac{i}{2} ( f(a, z) - f^*(a, z) ).
\end{align*}
\]
This system has a unique solution given by
\[
\begin{align*}
f(a, z) &= f^{[1]}(a, z) + if^{[i]}(a, z), \\
f^*(a, z) &= f^{[1]}(a, z) - if^{[i]}(a, z).
\end{align*}
\]
Using that \( f^{[\eta]} \) is real linear with respect to \( \eta \) (proven in lectures) we find that
\[
f^{[\eta]}(a, z) = f^{[1]}(a, z) \text{Re} \eta + f^{[i]}(a, z) \text{Im} \eta = \frac{1}{2} [ \overline{\eta} f(a, z) + \eta f^*(a, z) ],
\]
thus \( f, f^* \) indeed satisfy desired properties. Note that if \( f, f^* \) satisfy (1) for any \( \eta \) then we have \( f(a, z) = \partial/\partial \overline{\eta} f^{[\eta]}(a, z) \) and \( f^*(a, z) = \partial/\partial \eta f^{[\eta]}(a, z) \).
(b) Observe that
\[ \oint_{\partial \Omega} f^{[\eta]}(w, \zeta) f^{[\mu]}(z, \zeta) d\zeta = 2\pi i (\eta f(z, w) + \overline{\eta} f(w, z)). \]
Using that Im \( \oint_{\partial \Omega} f^{[\eta]}(w, \zeta) f^{[\mu]}(z, \zeta) d\zeta = 0 \) due to the boundary conditions we get the claim.

(c) Note that \( f(z, w) = \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} f^{[\eta,\mu]}(z, w) \), whereas \( f^*(z, w) = \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} f^{[\eta,\mu]}(z, w) \) and \( f^*(z, w) = \frac{\partial}{\partial \mu} \frac{\partial}{\partial \eta} f^{[\eta,\mu]}(z, w) \). Using these observations and (b) we immediately get the result.

(d) Consider the function \( f^{[\eta]}_{\Omega}((\phi(w), \phi'(z))/2) \). It satisfies the same properties as \( f^{[\eta]}(w, z) \), thus we have \( f^{[\eta]}_{\Omega}((\phi(w), \phi(w)) \cdot (\phi'(z))/2 = f^{[\eta]}_{\Omega}(w, z) \) due to the uniqueness of \( f^{[\eta]}_{\Omega}(w, z) \) and we can write
\[
\frac{1}{2} \left[ \eta f_{\Omega}(w, z) + \eta f_{\Omega}'(w, z) \right] = f^{[\eta]}_{\Omega}(w, z) = \]
\[
= f^{[\eta]}_{\Omega}((\phi(w), \phi(w)) \cdot (\phi'(z))/2 = \]
\[
= \frac{1}{2} \left[ \eta f_{\Omega}(\phi(w), \phi(z)) \cdot (\phi'(w))/2 + \eta f_{\Omega}((\phi(w), \phi(z)) \cdot (\phi'(w))/2 (\phi'(z))/2 \right].
\]
Using that \( f(a, z) \) and \( f^*(a, z) \) are uniquely defined we get the result

Recall that the holomorphic spinor \( g_{v,u}(z) \) (defined on the double cover of \( \Omega \) ramified over \( u, v \in \Omega, u \neq v \)) is uniquely characterized by the following conditions:
\[
g_{v,u}(z) = \frac{e^{-i \pi / 4}}{\sqrt{z-v}} \cdot [1 + O(z-v)] \quad \text{as} \quad z \to v;
\]
\[
g_{v,u}(z) = \frac{e^{i \pi / 4}}{\sqrt{z-u}} \cdot [c + O(z-u)] \quad \text{as} \quad z \to u, \quad \text{with an unknown} \quad c \in \mathbb{R},
\]
and the boundary conditions \( g_{v,u}(\zeta) \in (\tau(\zeta))^{-1/2} \mathbb{R} \) for \( \zeta \in \partial \Omega \). Further, recall that \( \mathcal{A}(v, u) \) is defined as the next coefficient in the expansion of \( g_{v,u}(z) \) as \( z \to v \):
\[
g_{v,u}(z) = \frac{e^{-i \pi / 4}}{\sqrt{z-v}} \cdot [1 + 2\mathcal{A}(v, u)(z-v) + O((z-v)^2)],
\]
and that
\[
\langle \sigma_u \sigma_v \rangle^+ := \exp \left[ \int \text{Re} \left[ \mathcal{A}(v, u)dv + \mathcal{A}(u, v)du \right] \right],
\]
where the multiplicative normalization is chosen so that \( \langle \sigma_u \sigma_v \rangle^+ \sim |u-v|^{-1/4} \) as \( u \to v \).

**Problem 2.** The goal is to prove the fusion rule \( \sigma \sigma \to 1 + \frac{\pi i}{2} + \ldots \), more precisely:
\[
\langle \sigma_u \sigma_v \rangle^+ = |v-u|^{-1/4} \cdot [1 + \frac{1}{2} \langle \epsilon_v \rangle^+ \cdot |v-u| + o(|v-u|)] \quad \text{as} \quad v \to u
\]
(not using explicit expressions available in simply connected \( \Omega \)), where the correlation functions \( \langle \sigma_u \sigma_v \rangle^+ \) and \( \langle \epsilon_v \rangle^+ \) are defined above.

Denote \( \eta := e^{i \pi / 4} \cdot (\nu - \overline{\nu})^{1/2}/|v-u|^{1/2} \). First, take for granted that \( \langle \xi_v \rangle^+ \to \langle \xi_u \rangle^+ \) and
\[
g_{v,u}(z) = |v-u|^{1/2} \cdot \left[ f^{[\eta]}(v, z) \cdot \left( \frac{z-v}{z-u} \right)^{1/2} + o(1) \right] \quad \text{as} \quad v \to u,
\]
uniformly on compact subsets \( z \in \Omega \setminus \{u\} \).

**Remark:** The right-hand side of (5) is chosen so that the difference does not blow up at \( z = v \) and approximately satisfies (3) and the boundary conditions, so it should be small.

(a) Deduce from (5) that

\[
2\mathcal{A}_{[v,u]} + \frac{1}{2(v-u)} = \langle \varepsilon_v \rangle^+_{\Omega} \cdot \frac{|v-u|}{v-u} + o(1) \quad \text{as} \quad v \to u.
\]

**Hint:** Consider \( \oint g_{[v,u]}(z) \cdot (z-u)^{1/2}(z-v)^{-3/2}dz \).

(b) Deduce (4) from (5) and the asymptotics \( \langle \sigma_u \sigma_v \rangle^+_{\Omega} \sim |v-u|^{-1/4} \) as \( v \to u \).

(c) Prove that \( \langle \varepsilon_v \rangle^+_{\Omega} \to \langle \varepsilon_u \rangle^+_{\Omega} \) as \( v \to u \).

**Hint:** Argue that each subsequential limit of \( f^{[n]}(v,\cdot) \) must coincide with \( f^{[n]}(u,\cdot) \).

(d)* Prove that (5).

**Solution.** We begin with the following observation. Let \( \overline{\Omega} \) be the domain in \( \mathbb{C} \) obtained by reflecting \( \Omega \) with respect to the real axis. Consider the functions \( \tilde{f} : \overline{\Omega} \times \Omega \to \mathbb{C} \) defined by \( \tilde{f}(v,z) = f^*(\overline{v},z) \). Notice that \( f^{[1]}(w,z) - i f^{[i]}(w,z) \) has a removable singularity at \( z = w \), hence \( \tilde{f} \) is defined on the whole \( \overline{\Omega} \times \Omega \). Due to the result of Problem 1(c) the function \( \tilde{f} \) is “separately” holomorphic, i.e. it is holomorphic in each variable when the other one is fixed. It follows from the Hartogs theorem that \( \tilde{f} \) is holomorphic as a function of two variables, hence we conclude that \( f^*(v,z) \) is analytic in \( \overline{v}, z \). In the same way we find that \( f(v,z) - \frac{2}{z-v} \) is holomorphic function in two variables. Notice that \( f(v,z) - \frac{2}{z-v} \) is antisymmetric, hence vanishes on the diagonal.

(a) Using the properties of \( f^{[n]} \) and the fact that \( f(v,z) = -f(z,v) \) we find that

\[
f^{[n]}(v,z) = e^{-i\frac{\pi}{4}}(v-u)^{1/2} \cdot \frac{1}{|v-u|^{1/2}(z-v)} + e^{i\frac{\pi}{4}} \cdot (\overline{v} - \overline{u})^{1/2}/|v-u|^{1/2} \langle \varepsilon_v \rangle^+_{\Omega} + O(z-v)
\]

as \( z \to v \). Substituting this relation and

\[
\left(\frac{z-v}{z-u}\right)^{1/2} = \frac{(z-v)^{1/2}}{(v-u)^{1/2}} \left(1 - \frac{z-v}{2(v-u)} + O(z-v)^2\right)
\]

into (5) we get the desired asymptotics.

(b) Due to the previous exercise we have

\[
\int \text{Re}[\mathcal{A}(v,u)dv + \mathcal{A}(u,v)du] = \int \text{Re}\left[-\frac{d(v-u)}{4(v-u)} + O(1)\right].
\]

The claim follows.

(c) As we mentioned above, the function \( f^*(v,z) \) is analytic in variables \( \overline{v}, z \) and therefore it is continuous. It follows that \( \langle \varepsilon_v \rangle^+_{\Omega} = f^*(v,v) \) is continuous too.
(d) Assume that \( u \) is fixed and \( u - v \) is small and consider the function
\[
F(z; v, u) := \left( g_{[v,u]}(z) - |v - u|^{1/2} \cdot f^{[v]}(v, z) \cdot \left( \frac{z - v}{z - u} \right)^{1/2} \right)^2.
\]

When \( z \) belongs to the boundary of \( \Omega \) we have
\[
\left( \frac{z - v}{z - u} \right)^{1/2} = 1 + O(v - u).
\]

Using the boundary conditions of \( g_{[v,u]} \) and \( f^{[v]}(v, z) \) we find that
\[
\int_{\partial \Omega} F(z; v, u) \, dz = \int_{\partial \Omega} |F(z; v, u)| \, |dz| + O(v - u)
\]

On the other hand, we have
\[
\int_{\partial \Omega} F(z; v, u) \, dz = 2\pi i \left( e^{i\pi/4} c - |v - u|^{1/2} f^{[v]}(v, u) \cdot (u - v)^{1/2} \right)^2.
\]

Using that
\[
f^{[v]}(v, u) = \frac{e^{i\pi/4} (u - v)^{1/2}}{|v - u|^{1/2} (u - v)} (1 + O(v - u))
\]
we find that
\[
\int_{\partial \Omega} F(z; v, u) \, dz = -2\pi (c - 1)^2 + O(v - u).
\]

Comparing these two expressions for the integral we get
\[
\int_{\partial \Omega} |F(z; v, u)| \, |dz| + 2\pi (c - 1)^2 = O(v - u).
\]

Now, let \( \Omega_{v,u} = \Omega \setminus \{z : |z - u| \leq |v - u|^2\} \). It follows that there exists a constant \( C > 0 \) such that
\[
\int_{\partial \Omega_{v,u}} |F(z; v, u)| \, |dz| \leq C|v - u|.
\]

Since \( F \) if holomorphic in \( \Omega_{v,u} \) we conclude that for any compact \( K \subset \Omega \setminus \{u\} \) there exists a constant \( C' \) such that such that
\[
\max_{z \in K} |F(z; v, u)| \leq C'|v - u|
\]
for provided \( |v - u| \) is small enough and (5) follows.

More information on correlations of \( \psi, \mu, \sigma, \varepsilon \) and fusion rules: [Section 4, arXiv:1605.09035]