Some notes for lectures by T. Sasamoto

1 GUE and Tracy-Widom distribution

The GUE (Gaussian unitary ensemble) of size $N \in \mathbb{N}$ is a random hermitian matrix H of size N with measure $\frac{1}{Z}e^{-\mathrm{Tr}H^2}$.

The joint eigenvalue $(x_i, 1 \le i \le N)$ density is explicitly given by

$$\frac{1}{Z}\prod_{1\le i<j\le N}(x_i-x_j)^2\prod_{i=1}^{N}e^{-x_i^2} = \frac{1}{Z}\det(x_i^{j-1})\det(x_i^{j-1})\prod_i e^{-x_i^2} = \det(\phi_{i-1}(x_j))\det(\phi_{i-1}(x_j)). \tag{1.1}$$

Here

$$\phi_n(x) = H_n(x)e^{-x^2/2}/c_n, \quad H_n\text{: Hermite polynomial, } c_n = \sqrt{\pi 2^n n!} \tag{1.2}$$

satisfies the orthonormality, $\int_{\mathbb{R}}\phi_n(x)\phi_m(x)dx = \delta_{nm}$. The second equality in (1.1) is due to the multi-linearity of a determinant. The normalization of (1.1) is easily checked by

$$\int_{\mathbb{R}^N}\det(\phi_i(x_j))\det(\psi_i(x_j))\prod_i dx_i = N!\det\left(\int_{\mathbb{R}}\phi_i(x)\psi_j(x)dx\right). \tag{1.3}$$

A measure in the form of a product of determinants as in (1.1) is related to a free fermion and a determinantal point process.

The distribution function of the largest eigenvalue $x_{\max}$ is written as

$$\begin{aligned}
&\mathbb{P}[x_{\max}\le s] \\
&= \int_{(-\infty,s]^N}\det(\phi_{i-1}(x_j))\det(\phi_{i-1}(x_j))\prod_i dx_i = \det\left(\int_{\mathbb{R}}\phi_i(x)(\phi_j(x)-\phi_j(x)1_{(s,\infty)})dx\right) \\
&= \det\left(\delta_{i,j}-\int_s^\infty \phi_{i-1}(x)\phi_{j-1}(x)dx\right) = \det(1-K)_{L^2(s,\infty)}
\end{aligned} \tag{1.4}$$

where the determinant on the rightmost side is the Fredholm determinant with kernel

$$K_N(x,y) = \sum_{n=0}^{N-1}\phi_n(x)\phi_n(y). \tag{1.5}$$

In (1.4), we use (1.3) in the second and $\det(1-AB)=\det(1-BA)$ in the last equalities.

By using an integral representation of the Hermite polynomial, one can show the asymptotics

$$\phi_{N-N^{1/3}\lambda}\left(\sqrt{2N}+\frac{\xi}{\sqrt{2}N^{1/6}}\right) \sim \frac{2^{1/4}}{N^{1/12}}\mathrm{Ai}(\xi+\lambda). \tag{1.6}$$

Using this we can establish

$$\lim_{N\to\infty}\mathbb{P}[(x_{\max}-\sqrt{2N})\sqrt{2}N^{1/6}\le s] = \det(1-K)_{L^2(s,\infty)} =: F_2(s) \tag{1.7}$$

where K is the Airy kernel,

$$K(\xi,\zeta) = \int_0^\infty \mathrm{Ai}(\xi+\lambda)\mathrm{Ai}(\zeta+\lambda)d\lambda. \tag{1.8}$$

$F_2(s)$ is the GUE Tracy-Widom distribution.