Solutions of exercises for Sasamoto's course. TA1

Problem 0.i
Recall that for any cylindric function f we have

$$\frac{d}{dt}\mathbb{E}[f] = \mathbb{E}[Lf],$$

with

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} \left(p\eta_j(1-\eta_{j+1}) + q\eta_{j+1}(1-\eta_j)\right)\left(f(\eta^{j,j+1}) - f(\eta)\right). \tag{1}$$

We now set $f(\eta) = \eta_k$ for a fixed integer k. With this choice the summation (1) reduces to

$$\begin{aligned} Lf(\eta) = &\Big(p\eta_k(1-\eta_{k+1}) + q\eta_{k+1}(1-\eta_k)\Big)(\eta_{k+1}-\eta_k) \\ &+ \Big(p\eta_{k-1}(1-\eta_k) + q\eta_k(1-\eta_{k-1})\Big)(\eta_{k-1}-\eta_k). \end{aligned}$$

Notice that

$$\Big(p\eta_k(1-\eta_{k+1}) + q\eta_{k+1}(1-\eta_k)\Big)(\eta_{k+1}-\eta_k) = \begin{cases} q\eta_{k+1}(1-\eta_k) & \text{if } \eta_{k+1}=1,\ \eta_k=0 \\ -p\eta_k(1-\eta_{k+1}) & \text{if } \eta_{k+1}=0,\ \eta_k=1 \\ 0 & \text{otherwise.} \end{cases}$$

Analogously

$$\Big(p\eta_{k-1}(1-\eta_k) + q\eta_k(1-\eta_{k-1})\Big)(\eta_{k-1}-\eta_k) = \begin{cases} -q\eta_k(1-\eta_{k-1}) & \text{if } \eta_k=1,\ \eta_{k-1}=0 \\ p\eta_{k-1}(1-\eta_k) & \text{if } \eta_k=0,\ \eta_{k-1}=1 \\ 0 & \text{otherwise.} \end{cases}$$

In conclusion we have

$$\frac{d}{dt}\mathbb{E}[\eta_k] = p\Big(\mathbb{E}[\eta_{k-1}(1-\eta_k)] - \mathbb{E}[\eta_k(1-\eta_{k+1})]\Big) + q\Big(\mathbb{E}[\eta_{k+1}(1-\eta_k)] - \mathbb{E}[\eta_k(1-\eta_{k-1})]\Big).$$

Problem 0.ii
Here we proof what is sometimes called the Andreief identity. Expanding the determinants $\det\{\phi_i(x_j)\}$ and $\det\{\psi_i(x_j)\}$ we can write their product as

$$\det\{\phi_i(x_j)\}\det\{\psi_i(x_j)\} = \sum_{\sigma,\tau\in S_N} \epsilon(\sigma)\epsilon(\tau)\prod_{i=1}^{N}\phi_{\sigma(i)}(x_i)\psi_{\tau(i)}(x_i).$$

Assuming the convergence of the integrals of functions ϕ, ψ we can exchange summations and integrals, obtaining

$$
\begin{aligned}
\int_{\mathbb{R}^N} \det\{\phi_i(x_j) \det\{\psi_i(x_j)\} \prod_i dx_i &= \sum_{\sigma,\tau \in S_N} \epsilon(\sigma)\epsilon(\tau) \prod_{i=1}^N \int_{\mathbb{R}} \phi_{\sigma(i)}(x)\psi_{\tau(i)}(x)\ dx \\
&= \sum_{\tau \in S_N} \sum_{\sigma \in S_N} \epsilon(\tau \circ \sigma)\epsilon(\sigma) \prod_{i=1}^N \int_{\mathbb{R}} \phi_{\sigma(i)}\psi_{\tau\circ\sigma(i)}(x)\ dx,
\end{aligned} \tag{2}
$$

where in the second equality we performed a change of summation index $\tau \to \tau \circ \sigma$. Since $\epsilon(\sigma \circ \tau)\epsilon(\sigma) = \epsilon(\tau)$ we see that each addend in the summation is constant in σ and therefore the right hand side of (2) becomes

$$
N! \sum_{\tau \in S_n} \epsilon(\tau) \prod_{i=1}^N \int_{\mathbb{R}} \phi_i(x)\psi_{\tau(i)}(x)dx = N! \det\left(\int_{\mathbb{R}} \phi_i(x)\psi_j(x)dx \right).
$$

Problem 1.i.a

The integrand in the expression of F_n only has poles at 0 and 1 and therefore the complex contour can be chosen to be a disk arbitrary large large radius R, that we denote with D_R

$$
F_{n+1}(x,t) = \frac{1}{2\pi i} \int_{D_R} dz \frac{1}{z^{x+1}} \left(1 - \frac{1}{z}\right)^{-n-1} \mathrm{e}^{-(1-z)t}. \tag{3}
$$

When $|z| > 1$ we can safely exchange the infinite summation sign with the integral sign and take the geometric summation obtaining

$$
\begin{aligned}
\sum_{y=x}^{\infty} F_n(y,t) =& \frac{1}{2\pi i} \int_{D_R} dz \left(\sum_{y=x}^{\infty} \frac{1}{z^{x+1}} \right) \left(1 - \frac{1}{z}\right)^{-n} \mathrm{e}^{-(1-z)t} \\
=& \frac{1}{2\pi i} \int_{D_R} dz \frac{1}{z^{x+1}} \left(1 - \frac{1}{z}\right)^{-n-1} \mathrm{e}^{-(1-z)t} \\
=& F_{n+1}(x,t).
\end{aligned} \tag{4}
$$

Problem 1.i.b

This simply follows by integrating the t dependent factor in the integrand function. We have

$$
\begin{aligned}
\int_0^t F_n(x,t')dt' =& \frac{1}{2\pi i} \int_{D_R} dz \frac{1}{z^{x+1}} \left(1 - \frac{1}{z}\right)^{-n} \left[- \frac{\mathrm{e}^{-(1-z)t'}}{(1-z)} \right]_{t'=0}^{t'=t} \\
=& \frac{1}{2\pi i} \int_{D_R} dz \frac{1}{z^{x+1}} \left(1 - \frac{1}{z}\right)^{-n-1} \left(1 - \mathrm{e}^{-(1-z)t}\right) \\
=& F_{n+1}(x,t) - F_{n+1}(x,0)
\end{aligned} \tag{5}
$$

Problem 1.ii

We write down the time derivative of the function G using the multilinearity property of the determinant

$$
\frac{d}{dt} G(x_1,x_2;t) = \begin{vmatrix} \frac{d}{dt} F_0(x_1 - y_1;t) & F_1(x_2 - y_1;t) \\ \frac{d}{dt} F_{-1}(x_1 - y_2;t) & F_0(x_2 - y_2;t) \end{vmatrix} + \begin{vmatrix} F_0(x_1 - y_1;t) & \frac{d}{dt} F_1(x_2 - y_1;t) \\ F_{-1}(x_1 - y_2;t) & \frac{d}{dt} F_0(x_2 - y_2;t) \end{vmatrix}. \tag{6}
$$

By using relations of the function F we have

$$\begin{aligned}\begin{vmatrix}\frac{d}{dt}F_0(x_1-y_1;t) & F_1(x_2-y_1;t)\\ \frac{d}{dt}F_{-1}(x_1-y_2;t) & F_0(x_2-y_2;t)\end{vmatrix} &= \begin{vmatrix}F_0(x_1-y_1-1;t)-F_0(x_1-y_1;t) & F_1(x_2-y_1;t)\\ F_{-1}(x_1-y_2-1;t)-F_{-1}(x_1-y_2;t) & F_0(x_2-y_2;t)\end{vmatrix}\\ &= G(x_1-1,x_2;t)-G(x_1,x_2;t).\end{aligned} \tag{7}$$

and analogously we write

$$\begin{vmatrix}F_0(x_1-y_1;t) & \frac{d}{dt}F_1(x_2-y_1;t)\\ F_{-1}(x_1-y_2;t) & \frac{d}{dt}F_0(x_2-y_2;t)\end{vmatrix} = G(x_1,x_2-1;t)-G(x_1,x_2;t). \tag{8}$$

To check the boundary conditions we use again recurrence relations of F. We write

$$\begin{aligned}G(x_1,x_1+1;t) &= \begin{vmatrix}F_0(x_1-y_1;t) & F_1(x_1-y_1+1;t)\\ F_{-1}(x_1-y_2;t) & F_0(x_1-y_2+1;t)\end{vmatrix}\\ &= \begin{vmatrix}F_0(x_1-y_1;t) & F_1(x_1-y_1;t)-F_0(x_1-y_1;t)\\ F_{-1}(x_1-y_2;t) & F_0(x_1-y_2;t)-F_{-1}(x_1-y_2;t)\end{vmatrix}\\ &= \begin{vmatrix}F_0(x_1-y_1;t) & F_1(x_1-y_1;t)\\ F_{-1}(x_1-y_2;t) & F_0(x_1-y_2;t)\end{vmatrix}\\ &= G(x_1,x_1;t).\end{aligned} \tag{9}$$

Finally we can verify the initial conditions. Through a residue computation we see that

$$G(x_1,x_2;0) = \begin{vmatrix}\mathbf{1}_{x_1=y_1} & \mathbf{1}_{y_1\geq x_2}\\ \mathbf{1}_{x_1=y_2}-\mathbf{1}_{x_1=y_2+1} & \mathbf{1}_{x_2=y_2}\end{vmatrix} = \mathbf{1}_{x_1=y_1}\mathbf{1}_{x_2=y_2}, \tag{10}$$

since $y_1 < y_2$ and the only region allowed for indices x_1, x_2 is $x_1 < x_2$.