Exercise problems for lectures by T. Sasamoto

1 Introduction

(i) Derive the time evolution equation for $\langle \eta_j \rangle$ by using the definition of the generator.
(ii) Give a proof to the identity

$$\int_{\mathbb{R}^N} \det(\phi_i(x_j)) \det(\psi_i(x_j)) \prod_i dx_i = N! \det\left(\int_{\mathbb{R}} \phi_i(x)\psi_j(x)dx \right).$$

1 TASEP

(i)Define

$$F_n(x,t) = \frac{1}{2\pi i} \int_{0,1} dz \frac{1}{z^{x+1}} (1 - 1/z)^{-n} e^{-(1-z)t}.$$

Check the following relations.
(a)

$$F_{n+1}(x,t) = \sum_{y=x}^{\infty} F_n(y,t) \tag{1.1}$$

(b)

$$\int_0^t F_n(x,t) = F_{n+1}(x+1,t) - F_{n+1}(x+1,0) \tag{1.2}$$

(ii) For $N = 2$, the Green's function reads

$$G(x_1,x_2;t) = \begin{vmatrix} F_0(x_1 - y_1;t) & F_1(x_2 - y_1;t) \\ F_{-1}(x_1 - y_2;t) & F_0(x_2 - y_2;t) \end{vmatrix}. \tag{1.3}$$

Check that this satisfies the following conditions it should satisfy.

$$\frac{d}{dt}G(x_1,x_2;t) = G(x_1 - 1,x_2;t) + G(x_1,x_2 - 1;t) - 2G(x_1,x_2;t), \tag{1.4}$$

$$G(x_1,x_1,t) = G(x_1,x_1 + 1;t), \tag{1.5}$$

$$G(x_1,x_2;t|y_1,y_2;0) = \delta_{x_1y_1}\delta_{x_2y_2}. \tag{1.6}$$

(ii*) Check the Green's function formula for general ASEP.
[Refs: Schütz J.Stat.Phys.88(1997)427, Tracy Widom CMP279(2008)815 (Errata CMP304(2011)875)]
(iii) Johansson showed the following formula, for TASEP with step i.c.

$$\mathbb{P}[N(t) \geq N] = \frac{1}{Z'_{N2}} \int_{[0,t]^N} \prod_{1\leq j<k\leq N} (x_j - x_k)^2 \prod_{j=1}^{N} e^{-x_j} dx_1 \dots dx_N. \tag{1.7}$$

Z'_{N2} is a normalization. Prove this for $N = 2$.