Exercise problems for lectures by T. Sasamoto

1 Introduction

(i) Derive the time evolution equation for \( \langle \eta_j \rangle \) by using the definition of the generator.
(ii) Give a proof to the identity

\[
\int_{\mathbb{R}^N} \det(\phi_i(x_j)) \det(\psi_i(x_j)) \prod_i dx_i = N! \det \left( \int_{\mathbb{R}} \phi_i(x) \psi_j(x) dx \right).
\]

1 TASEP

(i) Define

\[
F_n(x, t) = \frac{1}{2\pi i} \int_{0,1} dz \frac{1}{z^{x+1}} (1 - 1/z)^{-n} e^{-(1-z)t},
\]

Check the following relations.
(a) \( F_{n+1}(x, t) = \sum_{y=x}^{\infty} F_n(y, t) \) \( (1.1) \)

(b) \( \int_0^t F_n(x, t) = F_{n+1}(x+1, t) - F_{n+1}(x+1, 0) \) \( (1.2) \)

(ii) For \( N = 2 \), the Green’s function reads

\[
G(x_1, x_2; t) = \begin{vmatrix}
F_0(x_1 - y_1; t) & F_1(x_2 - y_1; t) \\
F_{-1}(x_1 - y_2; t) & F_0(x_2 - y_2; t)
\end{vmatrix}.
\]

Check that this satisfies the following conditions it should satisfy.

\[
\frac{d}{dt} G(x_1, x_2; t) = G(x_1 - 1, x_2; t) + G(x_1, x_2 - 1; t) - 2G(x_1, x_2; t), \quad (1.4)
\]

\[
G(x_1, x_1, t) = G(x_1, x_1 + 1; t), \quad (1.5)
\]

\[
G(x_1, x_2; t|y_1, y_2; 0) = \delta_{x_1 y_1} \delta_{x_2 y_2}. \quad (1.6)
\]

(ii*) Check the Green’s function formula for general ASEP.


(iii) Johansson showed the following formula, for TASEP with step i.c.

\[
\mathbb{P}[N(t) \geq N] = \frac{1}{Z_{N2}'} \int_{[0,t]^N} \prod_{1 \leq j<k \leq N} (x_j - x_k)^2 \prod_{j=1}^{N} e^{-x_j} dx_1 \ldots dx_N. \quad (1.7)
\]

\( Z_{N2}' \) is a normalization. Prove this for \( N = 2 \).