Exercise problems for lectures by T. Sasamoto

1 Introduction

(i) Derive the time evolution equation for $\langle \eta_j \rangle$ for ASEP by using the definition of the generator.
(ii) Give a proof to the identity

$$\int_{\mathbb{R}^N} \det(\phi_i(x_j)) \det(\psi_i(x_j)) \prod_i dx_i = N! \det\left(\int_{\mathbb{R}} \phi_i(x)\psi_j(x)dx \right).$$

1 TASEP

(i)Define

$$F_n(x,t) = \frac{1}{2\pi i} \int_{0,1} dz \frac{1}{z^{x+1}} (1-1/z)^{-n} e^{-(1-z)t}.$$

where $\int_{0,1}$ means a contour which encircles both $0,1$. Check the following relations.
(a)

$$F_{n+1}(x,t) = \sum_{y=x}^{\infty} F_n(y,t) \tag{1.1}$$

(b)

$$\int_0^t F_n(x,t) = F_{n+1}(x+1,t) - F_{n+1}(x+1,0) \tag{1.2}$$

(ii) For $N=2$, the Green's function reads

$$G(x_1,x_2;t) = \begin{vmatrix} F_0(x_1-y_1;t) & F_1(x_2-y_1;t) \\ F_{-1}(x_1-y_2;t) & F_0(x_2-y_2;t) \end{vmatrix}. \tag{1.3}$$

Check that this satisfies the following conditions it should satisfy.

$$\frac{d}{dt}G(x_1,x_2;t) = G(x_1-1,x_2;t) + G(x_1,x_2-1;t) - 2G(x_1,x_2;t), \tag{1.4}$$

$$G(x_1,x_1,t) = G(x_1,x_1+1;t), \tag{1.5}$$

$$G(x_1,x_2;t=0|y_1,y_2;0) = \delta_{x_1y_1}\delta_{x_2y_2}. \tag{1.6}$$

(ii*) Check the Green's function formula for general ASEP.
[Refs: Schütz J.Stat.Phys.88(1997)427, Tracy Widom CMP279(2008)815 (Errata CMP304(2011)875)]
(iii) Johansson showed the following formula, for TASEP with step i.c.

$$\mathbb{P}[N(t) \geq N] = \frac{1}{Z'_{N2}} \int_{[0,t]^N} \prod_{1\leq j<k\leq N} (x_j-x_k)^2 \prod_{j=1}^{N} e^{-x_j} dx_1 \dots dx_N. \tag{1.7}$$

Z'_{N2} is a normalization. Prove this for $N=2$.

(iii*) The arguments can be applied to general initial configuration. Show the following. Under the condition $M > y_N - y_1$,

$$\begin{aligned}
&\mathbb{P}[X_1(t) \geq y_1 + M] \\
&= \sum_{y_1+M \leq x_1 < x_2 < \cdots < x_N} G(x_1, x_2, \cdots, x_N; t | y_1, y_2, \cdots, y_N; 0) \\
&= \frac{1}{\prod_{j=1}^{N} j!} \int_{[0,t]^N} dt_1 \cdots dt_N \prod_{1 \leq j < k \leq N} (t_k - t_j) \det\left(F_{-j+1}(y_1 - y_j + M - 1; t_{N-k+1})\right). \quad (1.8)
\end{aligned}$$

[Ref: Nagao TS Nucl.Phys.B 699(2004)487]

2 Schur process and q-Whittaker process

(i)(a) Show, when $|\lambda| \leq n, z = (z_1, \cdots, z_n)$,

$$\sum_{j=1}^{n} \frac{1}{z_j} \times s_\lambda(z) = \sum_{j=1}^{n} s_{(\lambda_1, \cdots, \lambda_j - 1, \cdots, \lambda_n)}(z)$$

(b) Check that the Schur process ($N = 2$ case) of the form,

$$\frac{1}{Z} s_{\lambda^1}(1) s_{\lambda^2/\lambda^1}(1) s_{\lambda^2}(t), \quad Z = e^{2t}, \tag{2.1}$$

satisfies the block-push dynamics on the Gelfand-Tsetlin cone.
(i*) Find another dynamics satisfied by the Schur process above.
[Ref: Matveev Petrov: Ann. H. Poincare D 4(2017) 1]
(ii) (a) The q-Hermite polynomials are defined as

$$H_n(\cos\theta) = \sum_{k=0}^{n} \frac{(q;q)_n e^{i(n-2k)\theta}}{(q;q)_k (q;q)_{n-k}} \tag{2.2}$$

Check the equivalence between the $N = 2$ q-Whittaker function and the q-Hermite polynomials.
(b) The q-Mehler formula for q-Hermite polynomials reads

$$\sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q) H_n(\cos\phi|q)}{(q;q)_n} r^n = \frac{(r^2;q)_\infty}{(re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{-i(\theta-\phi)}, re^{-i(\theta+\phi)};q)_\infty} \tag{2.3}$$

Check that this q-Mehler formula for q-Hermite polynomials is equivalent to the Cauchy identity for $N = 2$ q-Whittaker function.
(iii) Check that the formula

$$\sum_{l \in \mathbb{Z}} \frac{z^l}{(\zeta q^l;q)_\infty} = \frac{\theta(\zeta z)(q;q)_\infty}{\theta(\zeta)(z;q)_\infty} \tag{2.4}$$

follows from the Ramanujan's bilateral sum formula.
(iv) Show that by taking an appropriate $q \to 0$ limit, the q-exp generating function $\langle \frac{1}{(\zeta q^X;q)_\infty} \rangle$ for a random variable X tends to the probability distribution $\mathbb{P}[X \geq m], m \in \mathbb{Z}$.
(v) Find a dynamics satisfied by the $N = 2$ q-Whittaker process by generalizing (i).