2 Some q-functions and q-formulas

The q-Pochhammer symbol, or the q shifted factorial. For $|q|<1$ and $n\in\mathbb{N}$,

$$(a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n),\quad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2.9}$$

The q-binomial theorem.

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty},\ |z|<1. \tag{2.10}$$

In particular the $a=0$ case appears in many applications.

Ramanujan's summation formula (cf [1] p502, [2] p138) is a two-sided generalization of the above q-binomial theorem (2.10). For $|q|<1, |b/a|<|z|<1$,

$$\sum_{n\in\mathbb{Z}}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(az;q)_\infty(\frac{q}{az};q)_\infty(q;q)_\infty(\frac{b}{a};q)_\infty}{(z;q)_\infty(\frac{q}{a};q)_\infty(b;q)_\infty(\frac{b}{az};q)_\infty}. \tag{2.11}$$

3 Frobenius determinant

Let us introduce a modified Jacobi theta function ([2] p303),

$$\theta(z)=(z;q)_\infty(q/z;q)_\infty,\ |q|<1, z\neq 0, \tag{3.12}$$

which is related to the ordinary theta function as

$$\theta_1(x,e^{\pi i\tau})=ie^{-ix+\pi i\tau/4}(q;q)_\infty\theta(e^{2ix};q),\quad q=e^{2\pi i\tau}, \operatorname{Im}\tau>0, x\in\mathbb{C}. \tag{3.13}$$

This function has been playing important role in various places. In the following we also use

$$\tilde{\theta}(z)=\frac{1}{\sqrt{z}}\theta(z), \tag{3.14}$$

which has the nice symmetry property, $\tilde{\theta}(1/z)=-\tilde{\theta}(z)$ where the square root is $\sqrt{z}=e^{\frac{1}{2}\log z}$ with the standard branch cut of logarithm.

Let $[x]$ be a nonzero holomorphic function which satisfies $[-x]=-[x]$ and the Riemann relation,

$$\begin{aligned}&[x+y][x-y][u+v][u-v]\\&=[x+u][x-u][y+v][y-v]-[x+v][x-v][y+u][y-u].\end{aligned} \tag{3.15}$$

It is known that $[x]$ satisfying the above two relations is necessarily in the form $e^{ax^2+b}f(cx)$ where $f(x)$ is either $f(x)=x$, $f(x)=\sin\pi x$ or $f(x)=\sigma(x)$ (cf [3] p451 20·53 ex.4 and p461 ex.38). Here the Weierstrass sigma function $\sigma(x)=\sigma(x|\omega_1,\omega_2)$, with the half periods ω_1,ω_2, can be written in terms of the ordinary theta function as (cf [3] p473 21·43)

$$\sigma(x|\omega_1,\omega_2)=\frac{2\omega_1}{\pi\theta_1^{(1)}}\exp\left(-\frac{\pi^2x^2\theta_1^{(3)}}{24\omega_1^2\theta_1^{(1)}}\right)\theta_1(\frac{\pi x}{2\omega_1},e^{i\pi\frac{\omega_2}{\omega_1}}) \tag{3.16}$$

where $\theta_1^{(n)} = \frac{d^n}{dx^n}\theta_1(x,q)|_{x=0}, n \in \mathbb{N}$. Combining (3.13), (3.16), one sees that our theta function $\tilde{\theta}(q^x)$ is written in the form $e^{ax^2+b}\sigma(cx)$ and hence an example of $[x]$.

Theorem (Frobenius 1882) For $[x]$ as above, the following Cauchy determinant formula holds,

$$\frac{[\nu + B - C]\prod_{1\le i<j\le N}[b_i - b_j][c_j - c_i]}{[\nu]\prod_{i,j=1}^N [b_i - c_j]} = \det\left(\frac{[\nu + b_i - c_j]}{[\nu][b_i - c_j]}\right)_{1\le i,j\le N}$$

where ν is a parameter, $b_i, c_i, 1 \le i \le N$ are $2N$ complex variables and $B = \sum_{i=1}^N b_i, C = \sum_{i=1}^N c_i$.

For the theta function $\tilde{\theta}$ this reads

$$\frac{\tilde{\theta}(\frac{\zeta A}{Z})\prod_{i<j}\tilde{\theta}(a_i/a_j)\prod_{i<j}\tilde{\theta}(z_i/z_j)}{\tilde{\theta}(\zeta)\prod_{i,j}\tilde{\theta}(a_i/z_j)} = \det\left(\frac{\tilde{\theta}(\zeta a_i/z_j)}{\tilde{\theta}(\zeta)\tilde{\theta}(a_i/z_j)}\right) \tag{3.17}$$

with $A = \prod_i A_i, Z = \prod_i z_i$.

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions. Volume 71 of Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 1999.

[2] G. Gasper, M. Rahman, *Basic hypergeometric series*, Cambridge, 2004.

[3] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge, 1927.