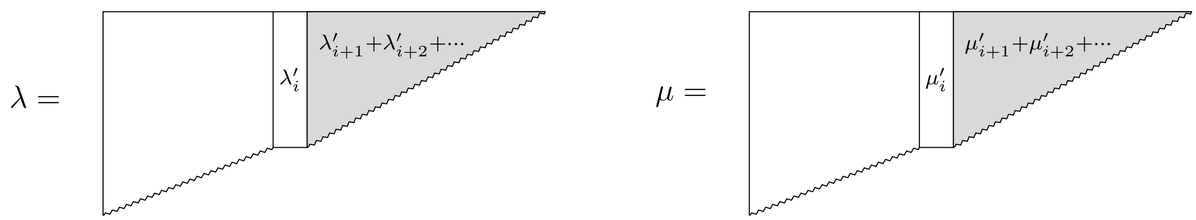
1. Dominance order. Recall that the dominance order ($\geqslant$) on the set $\{\lambda \vdash n\}$ of partitions of size n is defined by $\lambda \geqslant \mu$ if $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$ for all $i \geqslant 1$. This is a total order for $n \leqslant 5$ and a partial order for all $n \geqslant 6$. Show that $\lambda \geqslant \mu$ if and only if $\lambda' \leqslant \mu'$.

Solution. Proceeding by contradiction, assume that $\lambda \geqslant \mu$ but $\lambda' \nleqslant \mu'$. Then there exists a smallest positive integer i such that $\lambda'_1 + \cdots + \lambda'_i > \mu'_1 + \cdots + \mu'_i$. To pass from this inequality for the area of the first i columns of λ and μ to an equality for the rows, notice that since $|\lambda| = |\mu|$ we also have

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots < \mu'_{i+1} + \mu'_{i+2} + \cdots . \tag{1}$$

As is easily seen from the figure



it follows that

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots = \sum_{k=1}^{\lambda'_i} (\lambda_k - i) \qquad \text{and} \qquad \mu'_{i+1} + \mu'_{i+2} + \cdots = \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

Hence (1) can also be expressed as

$$\sum_{k=1}^{\lambda'_i} (\lambda_k - i) < \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

The minimality of i implies that $\lambda'_i > \mu'_i$, so that

$$\sum_{k=1}^{\mu'_i} (\lambda_k - i) < \sum_{k=1}^{\lambda'_i} (\lambda_k - i) < \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

Including the $i \times \mu'_i$ rectangle on both sides, we obtain the following inequality for the area of the first μ'_i rows of λ and μ

$$\sum_{k=1}^{\mu'_i} \lambda_k < \sum_{k=1}^{\mu'_i} \mu_k.$$

This contradicts the fact that $\lambda \geqslant \mu$.

2. The centralizer of the symmetric group. Show that

$$z_\lambda := \prod_{i \geqslant 1} i^{m_i}\, m_i! = |Z_w|,$$

where $m_i = m_i(\lambda)$ is the multiplicity of i (the number of parts equal to i) in λ, $w \in S_n$ has cycle type λ (i.e., w has m_i cycles of length i), and Z_w is the centralizer of w.

Solution. Conjugation of elements of S_n does not change their cycle type (the conjugacy classes of S_n are formed by the permutations of the same cycle type). Specifically, if

$$w = (c_1, \dots, c_{k_1})(c_{k_1+1}, \dots, c_{k_2}) \dots (c_{k_{r-1}+1}, \dots, c_{k_r})$$

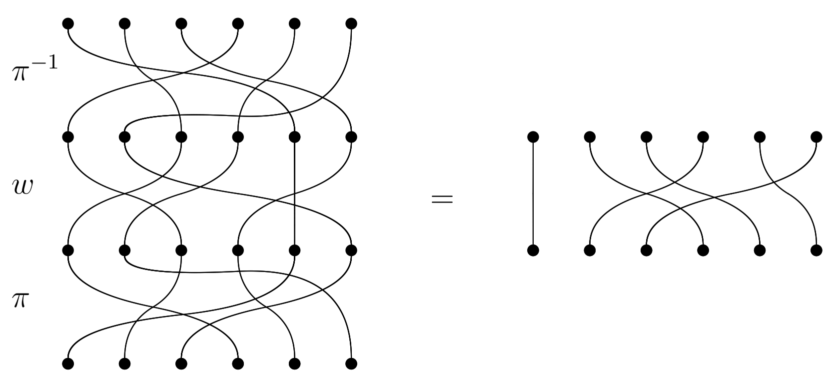
(where $k_r := n$) then

$$\pi w \pi^{-1} = (\pi_{c_1}, \dots, \pi_{c_{k_1}})(\pi_{c_{k_1+1}}, \dots, \pi_{c_{k_2}}) \dots (\pi_{c_{k_{r-1}+1}}, \dots, \pi_{c_{k_r}}).$$

For example, if $w = (13)(264)(5) \in S_6$ (in one-line notation $w = (361254)$) and $\pi = (145)(263)$ (in one-line notation $\pi = (462513)$), then $\pi^{-1} = (154)(236)$ (in one-line notation $\pi^{-1} = (536142)$) and thus

$$\begin{aligned} \pi w \pi^{-1} &= (\pi_1, \pi_3)(\pi_2, \pi_6, \pi_4)(\pi_5) \\ &= (42)(635)(1) \\ &= (1)(24)(356) \end{aligned}$$

(in one-line notation $\pi w \pi^{-1} = (415263)$). Pictorially



For π to be in Z_w we require $\pi w \pi^{-1} = w$. In other words, π can permute the cycles (of w) of fixed length and/or cycle each cycle as in $(abc \dots z) \mapsto (rs \dots zab \dots q)$. If there are m_i cycles of length i in w it thus follows that

$$|Z_w| = \prod_{i \geqslant 1} i^{m_i} m_i!.$$

3. Gaussian polynomials. Let n, m be nonnegative integers. Then the Gaussian polynomials or q-binomial coefficients are defined as

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\lambda \subseteq (m^n)} q^{|\lambda|}. \tag{2}$$

Here the sum is over all partitions λ that fit in a rectangle of height n and width m, i.e., partitions λ such that $\lambda_1 \leqslant m$ and $l(\lambda) \leqslant n$.

(a) Show that
   (i) $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ (initial condition);

   (ii) $\begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{bmatrix} n+m \\ n \end{bmatrix}$ (symmetry);

   (iii) $\begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{bmatrix} n+m-1 \\ m \end{bmatrix} + q^n \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix} = q^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix} + \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$ (q-Pascal identities).

(b) Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{k=0}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}. \tag{3}$$

Remark. One similarly shows the q-Chu–Vandermonde or Durfee rectangle identity

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\ell=0}^{n} q^{\ell(\ell+k)} \begin{bmatrix} m-k \\ \ell \end{bmatrix} \begin{bmatrix} n+k \\ \ell+k \end{bmatrix},$$

where k is an arbitrary integer, and $k = 0$ corresponds to the Durfee square identity. (The Durfee square of a partition is the largest square contained in its diagram.)

(c) Use (a) to construct the first six rows of the q-Pascal triangle and check that $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ computed this way matches the definition (2).

(d) Let $(a;q)_n := (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ denote a q-shifted factorial or q-Pochhammer symbol. Use (a) to show that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^{n-m+1};q)_m}{(q;q)_m}.$$

(e) Let $[n] := (1-q^n)/(1-q) = 1 + q + \cdots + q^{n-1}$ and $[n]! := [1][2]\ldots[n]$. Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \frac{[n+m]!}{[n]!\,[m]!} \quad \text{and} \quad \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}.$$

Solution.

(a) (i) The only partition contained in a rectangle of zero width is the empty partition 0.
(ii) Replace the summation index $\lambda \in (n^m)$ by $\lambda' \in (m^n)$ and use $|\lambda| = |\lambda'|$.
(iii) For the first recursion, we dissect the sum (2) according to the length of λ. The term $\begin{bmatrix} n+m-1 \\ m \end{bmatrix}$ corresponds to the generating function of partitions of length at most $n-1$ (those partitions fit in an $(n-1) \times m$ rectangle) and the term $q^n \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$

corresponds to the generating function of partitions of length exactly n. Since the diagram of such partitions has a first column of height n (contributing q^n to the generating function), after stripping off this column we are left with a partition that fits in an $n \times (m-1)$ rectangle, which contributes $\begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$ to the generating function. The second recursion follows from (ii) or by carrying out a dissection according to $\lambda_1 < m$ and $\lambda_1 = m$.

(b) Note that part (iii) of (a) holds for all integers m if we define $\begin{bmatrix} n+m \\ m \end{bmatrix} := 0$ for m a negative integer. To obtain (3) we may either iterate the second recursion in (iii), which implies that

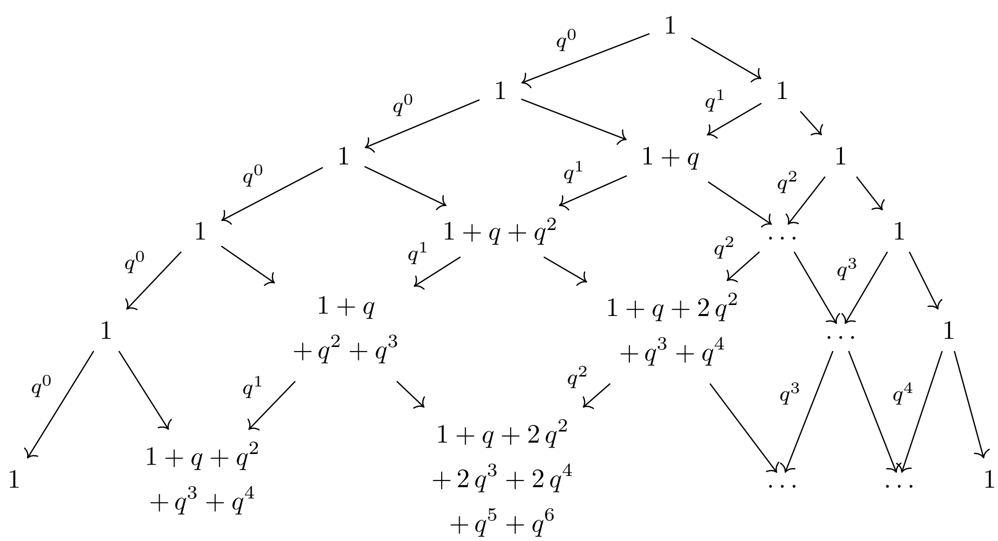
$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{bmatrix} n+m-K-1 \\ m-K-1 \end{bmatrix} + \sum_{k=m-K}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}$$

for all nonnegative integers K. (For $K=-1$ the above is also true but tautological.) Choosing $K = m$ yields (3). Equivalently, conditioning on the value of λ_1 we may write the sum in (2) as

$$\sum_{k=0}^{m} \sum_{\substack{\lambda \subseteq (m^n) \\ \lambda_1 = k}} q^{|\lambda|} = \sum_{k=0}^{m} q^k \sum_{\lambda \subseteq (k^{n-1})} q^{|\lambda|} = \sum_{k=0}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}.$$

(For the Durfee rectangle identity fix k and similarly condition on the largest rectangle $((\ell+k)^\ell) \subseteq (m^n)$ that fits in λ.)

(c) Use part (iii) of (a) to construct rows $n+m = 0, 1, \ldots, 5$. Let us label the left-pointing arrow by the factor q^n and omit the factor 1 for the arrows going to the right:



(d) It suffices to check that the right-hand side with $n \mapsto n+m$ obeys the initial condition (i) and the q-Pascal recurrence (iii) from (a).

(e) This follows in a straightforward manner from (d).