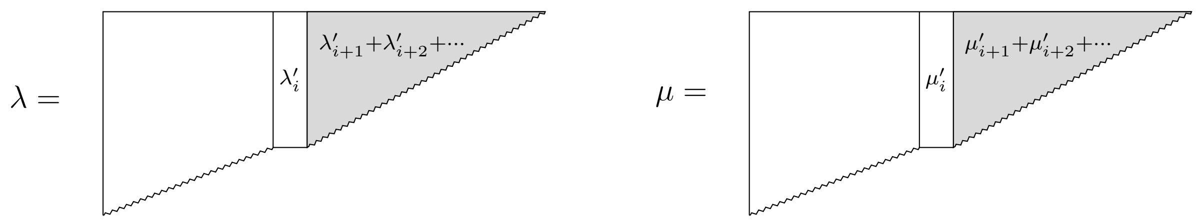
Solutions to exercises 1–3

1. Dominance order. Recall that the dominance order ($\geqslant$) on the set $\{\lambda \vdash n\}$ of partitions of size n is defined by $\lambda \geqslant \mu$ if $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$ for all $i \geqslant 1$. This is a total order for $n \leqslant 5$ and a partial order for all $n \geqslant 6$. Show that $\lambda \geqslant \mu$ if and only if $\lambda' \leqslant \mu'$.

Solution. Proceeding by contradiction, assume that $\lambda \geqslant \mu$ but $\lambda' \nleqslant \mu'$. Then there exists a smallest positive integer i such that $\lambda'_1 + \cdots + \lambda'_i > \mu'_1 + \cdots + \mu'_i$. To pass from this inequality for the area of the first i columns of λ and μ to an equality for the rows, notice that since $|\lambda| = |\mu|$ we also have

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots < \mu'_{i+1} + \mu'_{i+2} + \cdots . \tag{1}$$

As is easily seen from the figure



it follows that

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots = \sum_{k=1}^{\lambda'_i} (\lambda_k - i) \qquad \text{and} \qquad \mu'_{i+1} + \mu'_{i+2} + \cdots = \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

Hence (1) can also be expressed as

$$\sum_{k=1}^{\lambda'_i} (\lambda_k - i) < \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

The minimality of i implies that $\lambda'_i > \mu'_i$, so that

$$\sum_{k=1}^{\mu'_i} (\lambda_k - i) < \sum_{k=1}^{\lambda'_i} (\lambda_k - i) < \sum_{k=1}^{\mu'_i} (\mu_k - i).$$

Including the $i \times \mu'_i$ rectangle on both sides, we obtain the following inequality for the area of the first μ'_i rows of λ and μ

$$\sum_{k=1}^{\mu'_i} \lambda_k < \sum_{k=1}^{\mu'_i} \mu_k.$$

This contradicts the fact that $\lambda \geqslant \mu$.

2. The centralizer of the symmetric group. Show that

$$z_\lambda := \prod_{i\geqslant 1} i^{m_i}\, m_i! = |Z_w|,$$

where $m_i = m_i(\lambda)$ is the multiplicity of i (the number of parts equal to i) in λ, $w \in S_n$ has cycle type λ (i.e., w has m_i cycles of length i), and Z_w is the centralizer of w.

Solution. Conjugation of elements of S_n does not change their cycle type (the conjugacy classes of S_n are formed by the permutations of the same cycle type). Specifically, if

$$w = (c_1, \ldots, c_{k_1})(c_{k_1+1}, \ldots, c_{k_2}) \ldots (c_{k_{r-1}+1}, \ldots, c_{k_r})$$

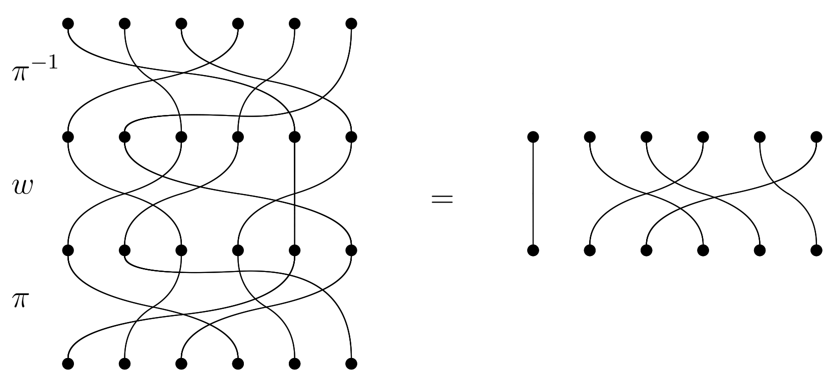
(where $k_r := n$) then

$$\pi w \pi^{-1} = (\pi_{c_1}, \ldots, \pi_{c_{k_1}})(\pi_{c_{k_1+1}}, \ldots, \pi_{c_{k_2}}) \ldots (\pi_{c_{k_{r-1}+1}}, \ldots, \pi_{c_{k_r}}).$$

For example, if $w = (13)(264)(5) \in S_6$ (in one-line notation $w = (361254)$) and $\pi = (145)(263)$ (in one-line notation $\pi = (462513)$), then $\pi^{-1} = (154)(236)$ (in one-line notation $\pi^{-1} = (536142)$) and thus

$$\begin{aligned}\pi w \pi^{-1} &= (\pi_1, \pi_3)(\pi_2, \pi_6, \pi_4)(\pi_5)\\ &= (42)(635)(1)\\ &= (1)(24)(356)\end{aligned}$$

(in one-line notation $\pi w \pi^{-1} = (415263)$). Pictorially



For π to be in Z_w we require $\pi w \pi^{-1} = w$. In other words, π can permute the cycles (of w) of fixed length and/or cycle each cycle as in $(abc \ldots z) \mapsto (rs \ldots zab \ldots q)$. If there are m_i cycles of length i in w it thus follows that

$$|Z_w| = \prod_{i\geqslant 1} i^{m_i} m_i!.$$

3. Gaussian polynomials. Let n, m be nonnegative integers. Then the Gaussian polynomials or q-binomial coefficients are defined as

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\lambda \subseteq (m^n)} q^{|\lambda|}. \tag{2}$$

Here the sum is over all partitions λ that fit in a rectangle of height n and width m, i.e., partitions λ such that $\lambda_1 \leqslant m$ and $l(\lambda) \leqslant n$.

(a) Show that
  (i) $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1$ (initial condition);
  (ii) $\left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} n+m \\ n \end{smallmatrix}\right]$ (symmetry);
  (iii) $\left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right] + q^n \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right] = q^m \left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right] + \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right]$ (q-Pascal identities).

(b) Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{k=0}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}. \tag{3}$$

Remark. One similarly shows the q-Chu–Vandermonde or Durfee rectangle identity

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\ell=0}^{n} q^{\ell(\ell+k)} \begin{bmatrix} m-k \\ \ell \end{bmatrix} \begin{bmatrix} n+k \\ \ell+k \end{bmatrix},$$

where k is an arbitrary integer, and $k = 0$ corresponds to the Durfee square identity. (The Durfee square of a partition is the largest square contained in its diagram.)

(c) Use (a) to construct the first six rows of the q-Pascal triangle and check that $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]$ computed this way matches the definition (2).

(d) Let $(a; q)_n := (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ denote a q-shifted factorial or q-Pochhammer symbol. Use (a) to show that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^{n-m+1}; q)_m}{(q; q)_m}.$$

(e) Let $[n] := (1-q^n)/(1-q) = 1 + q + \cdots + q^{n-1}$ and $[n]! := [1][2]\ldots[n]$. Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \frac{[n+m]!}{[n]!\,[m]!} \quad \text{and} \quad \lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}.$$

Solution.

(a) (i) The only partition contained in a rectangle of zero width is the empty partition 0.
(ii) Replace the summation index $\lambda \in (n^m)$ by $\lambda' \in (m^n)$ and use $|\lambda| = |\lambda'|$.
(iii) For the first recursion, we dissect the sum (2) according to the length of λ. The term $\left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right]$ corresponds to the generating function of partitions of length at most $n-1$ (those partitions fit in an $(n-1) \times m$ rectangle) and the term $q^n \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right]$

corresponds to the generating function of partitions of length exactly n. Since the diagram of such partitions has a first column of height n (contributing q^n to the generating function), after stripping off this column we are left with a partition that fits in an $n\times(m-1)$ rectangle, which contributes $\left[{n+m-1 \atop m-1}\right]$ to the generating function. The second recursion follows from (ii) or by carrying out a dissection according to $\lambda_1 < m$ and $\lambda_1 = m$.

(b) Note that part (iii) of (a) holds for all integers m if we define $\left[{n+m \atop m}\right] := 0$ for m a negative integer. To obtain (3) we may either iterate the second recursion in (iii), which implies that

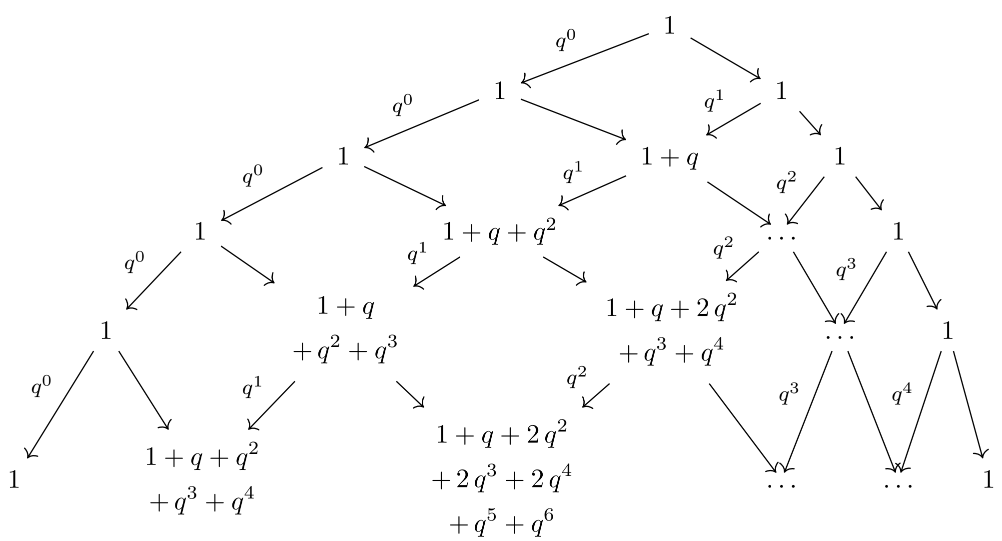
$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \begin{bmatrix} n+m-K-1 \\ m-K-1 \end{bmatrix} + \sum_{k=m-K}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}$$

for all nonnegative integers K. (For $K=-1$ the above is also true but tautological.) Choosing $K=m$ yields (3). Equivalently, conditioning on the value of λ_1 we may write the sum in (2) as

$$\sum_{k=0}^{m} \sum_{\substack{\lambda\subseteq(m^n) \\ \lambda_1=k}} q^{|\lambda|} = \sum_{k=0}^{m} q^k \sum_{\lambda\subseteq(k^{n-1})} q^{|\lambda|} = \sum_{k=0}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}.$$

(For the Durfee rectangle identity fix k and similarly condition on the largest rectangle $((\ell+k)^{\ell}) \subseteq (m^n)$ that fits in λ.)

(c) Use part (iii) of (a) to construct rows $n+m = 0, 1, \ldots, 5$. Let us label the left-pointing arrow by the factor q^n and omit the factor 1 for the arrows going to the right:



(d) It suffices to check that the right-hand side with $n \mapsto n+m$ obeys the initial condition (i) and the q-Pascal recurrence (iii) from (a).

(e) This follows in a straightforward manner from (d).

4. Plethystic notation. The aim of this question is to prove the q-binomial theorem

$$\sum_{k\geqslant 0} \frac{(a;q)_k}{(q;q)_k}\, z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{4}$$

using symmetric functions and plethystic notation. There are many alternative proofs, some of which are simpler, but hopefully this demonstrates the power of plethystic manipulations.

(a) To get a better feel for (4) show that, for n a nonnegative integer, it implies

(i) the q-binomial expansion

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} w^{n-k} z^k = \prod_{i=0}^{n-1} (w + q^i z);$$

(ii)

$$1 + \sum_{k\geqslant 1} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} z^k = \frac{1}{(z;q)_n}.$$

Remark. One can use (i) to prove Jacobi's triple product identity

$$\sum_{k=-\infty}^{\infty} (-z)^k q^{\binom{k}{2}} = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

by replacing $n \mapsto 2n$ followed by $k \mapsto k+n$ and using

$$(q^{-n}z;q)_{2n} = q^{-\binom{n+1}{2}} (-z)^n (z;q)_n (q/z;q)_n.$$

The triple product identity plays a key role in the theory of elliptic functions, and is the simplest example of a denominator identity for affine Kac–Moody Lie algebras, corresponding to $\mathrm{A}_1^{(1)}$ (affine $\mathfrak{sl}_2$).

(b) To prepare for the proof of (4) use plethystic notation to show that the generating function $\sigma_z[X] := \sum_{k\geqslant 0} h_k[X]\, z^k$ satisfies

(i) $\sigma_z[X+Y] = \sigma_z[X]\,\sigma_z[Y]$ and thus $h_k[X+Y] = \sum_{i=0}^{k} h_i[X]\, h_{k-i}[Y]$;

(ii) $\sigma_z[1] = \frac{1}{1-z}$ so $h_n[1] = 1$;

(iii) $\sigma_z[\frac{a}{1-q}] = \frac{1}{(az;q)_\infty}$.

(c) Now prove (4) by showing that both sides are equal to $\sigma_z[\frac{1-a}{1-q}]$.
Hint. For the left-hand side of (4) argue that it suffices to check equality with $\sigma_z[\frac{1-a}{1-q}]$ for a number of suitably chosen values of a and use part (b) to manipulate the alphabet on which h_r acts to recognise (3).

Solution.

(a) (i) Replace $(a,z)\mapsto(q^{-n},zq^n/w)$ where n is a nonnegative integer. Multiplying both sides of the resulting identity by w^n, the result follows since

$$\frac{(q^{-n};q)_k}{(q;q)_k}=q^{\binom{k}{2}+nk}\begin{bmatrix}n\\k\end{bmatrix}.$$

(ii) Replace $a\mapsto q^n$ and use that $(q^n;q)_k/(q;q)_k$ is 1 for $k=0$ and $\left[\begin{smallmatrix}k+n-1\\k\end{smallmatrix}\right]$ for $k>0$.

(b) (i) Since $\log\sigma_z[X]=\psi_z[X]:=\sum_{r\geqslant 1}\frac{p_r[X]}{r}\,z^r$ we have

$$\sigma_z[X+Y]=\mathrm{e}^{\psi_z[X+Y]}=\mathrm{e}^{\psi_z[X]+\psi_z[Y]}=\sigma_z[X]\sigma_z[Y].$$

Equating coefficients of z^n the convolution formula for h_k follows.
(ii) Note that for $X=\sum_{i\geqslant 1}x_i$,

$$\sigma_z[X]=\prod_{i\geqslant 1}\frac{1}{1-zx_i}.$$

Hence, if $X=x$ is a single-letter alphabet $\sigma_z[x]=1/(1-zx)$. The special case were this letter is 1 gives (ii).
(iii) Recall that $1/(1-q)=1+q+\cdots$. Hence

$$\sigma_z\Big[\frac{a}{1-q}\Big]=\prod_{i\geqslant 1}\frac{1}{1-azq^{i-1}}=\frac{1}{(az;q)_\infty}.$$

(c) To prove the q-binomial theorem (4) we first note that, since $\sigma_z[X-Y]=\sigma_z[X]/\sigma_z[Y]$,

$$\frac{(az;q)_\infty}{(z;q)_\infty}=\sigma_z\Big[\frac{1}{1-q}-\frac{a}{1-q}\Big]=\sigma_z\Big[\frac{1-a}{1-q}\Big].$$

The q-binomial theorem is thus equivalent to

$$\sum_{k\geqslant 0}\frac{(a;q)_k}{(q;q)_k}\,z^k=\sigma_z\Big[\frac{1-a}{1-q}\Big].$$

Equating coefficients of z^k yields

$$\frac{(a;q)_k}{(q;q)_k}=h_k\Big[\frac{1-a}{1-q}\Big],$$

where k is an arbitrary nonnegative integer. Since both sides are polynomials in a of degree k, it suffices to prove the above for $a=q^n$, where n is an arbitrary nonnegative integer. That is, we must prove

$$\frac{(q^n;q)_k}{(q;q)_k}=h_k\Big[\frac{1-q^n}{1-q}\Big]=h_k(1,q,\dots,q^n).$$

For $n = 0$ this is obvious so we may assume that $n \geqslant 1$ in the remainder. Then

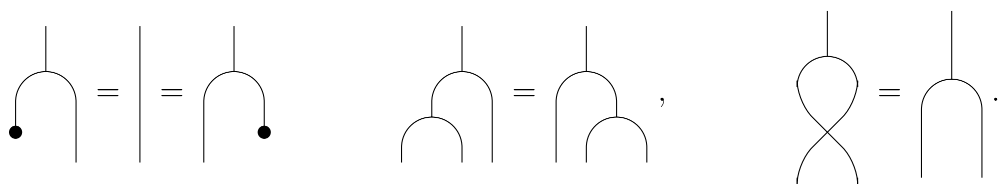
$$\begin{aligned} h_k\left[\frac{1-q^n}{1-q}\right] &= h_k\left[1 + q\,\frac{1-q^{n-1}}{1-q}\right] \\ &= \sum_{i=0}^{k} q^i h_i\left[\frac{1-q^{n-1}}{1-q}\right] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix} = \frac{(q^n;q)_k}{(q;q)_k}. \end{aligned}$$

Here we recognised the sum (3) which, together with the initial condition at $k = 0$, characterises the Gaussian polynomials.

5. The Hopf-algebra structure of Λ. In this exercise we examine the algebraic structure of the ring of symmetric functions. All tensor products are over $K \coloneqq \mathbb{Z}$.

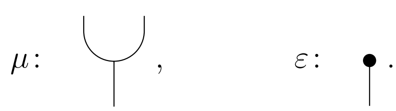
(a) Show that Λ is a commutative (unital associative) K-algebra with the usual product $m\colon \Lambda\otimes\Lambda \longrightarrow \Lambda$, $f[X]\,g[Y] \longmapsto f[X]\,g[X]$ in plethystic notation, and unit $e\colon K \longrightarrow \Lambda$ defined by $1 \longmapsto 1[X] \equiv 1$ extended K-linearly.
This structure can be nicely captured using string diagrams. Think of 'time' as increasing upwards, and depict (note that K is not drawn)

with $\gamma\colon \Lambda\otimes\Lambda \longrightarrow \Lambda\otimes\Lambda$ the permutation $f[X]\,g[Y] \longmapsto f[Y]\,g[X] = g[X]\,f[Y]$. The axioms of a (unital associative) algebra and commutativity then take the form

(5) 

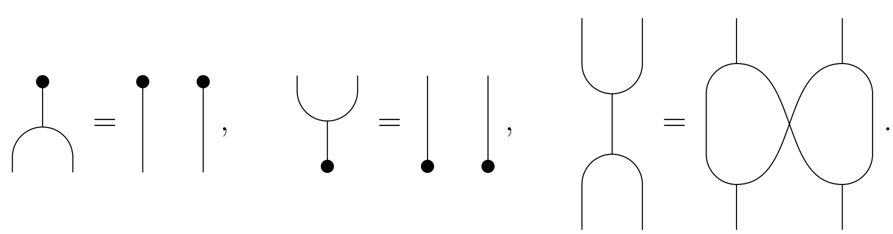
For example, the left-most diagram encodes $\Lambda \cong \Lambda\otimes K \xrightarrow{\mathrm{id}\otimes e} \Lambda\otimes\Lambda \xrightarrow{m} \Lambda$. Such an equality of string diagrams is often alternatively expressed as a commutative diagram.

(b) Show that Λ also is a cocommutative (counital coassociative) coalgebra, with coproduct $\mu\colon \Lambda \longrightarrow \Lambda\otimes\Lambda$ given by $f[X] \longmapsto f[X+Y]$ and counit $\varepsilon\colon \Lambda \longrightarrow K$, $f[X] \longmapsto f[0]$. Here the axioms are given by flipping all diagrams in (5) upside down and interpreting

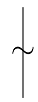
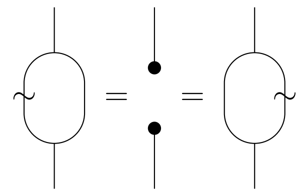


Do this using plethystic notation as well as by explicit computations for the power-sum basis p_λ.
Remark. Elements $f\in\Lambda$ whose coproduct satisfies $\mu(f) = f\otimes 1 + 1\otimes f$, such as the power sums p_r, are called primitive.

(c) Show that the preceding structures are compatible, making Λ a bialgebra: μ and ε are algebra homomorphisms (equivalently, m and e are coalgebra morphisms),



(d) Show that the bialgebra-structure of Λ extends to that of a Hopf algebra, with antipode $S\colon \Lambda \longrightarrow \Lambda$, $f[X] \longmapsto f[-X]$ extended as an (anti)homomorphism. Here the latter, depicted, say, as

S:  , must obey  .

Again do this using plethystic notation as well as by explicit calculation on the power-sum basis p_λ.

Remark. Viewed as equipped with this structure, Λ is commonly denoted by **Symm**.

Remark. Quantum integrability is related to quantum groups, which are sometimes defined as Hopf algebras that are quasitriangular, i.e., cocommutative up to conjugation by an invertible element of the tensor product of the Hopf algebra with itself, called the R-matrix, that behaves in a certain nice way under $\mu \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mu$ to guarantee it obeys the Yang–Baxter equation. Cocommutative Hopf algebras, such as Λ, are either viewed as boring examples ($R = 1 \otimes 1$) or excluded from the definition of a quantum group.

(e) Show that Λ is self dual with respect to the scalar product on $\Lambda \otimes \Lambda$ given by $\langle f_1[X]\, g_1[Y], f_2[X]\, g_2[Y] \rangle := \langle f_1[X], f_2[X] \rangle \langle g_1[Y], g_2[Y] \rangle$ where the right-hand side features the Hall scalar product on Λ. That is, the algebra and coalgebra structure of Λ are dual in the sense that $\langle f[X+Y], g[X]\, h[Y] \rangle = \langle f[X], g[X]\, h[X] \rangle$ and $\langle f[0], n \rangle = \langle f[X], n\, 1[X] \rangle$, where the scalar product on K is given by multiplication.

Solution.

(a) These are just the properties of the usual product.

(b) For the counit,

$$(\varepsilon \otimes \mathrm{id})\, f[X+Y] = f[0+Y] = f[Y] \equiv f[X] = f[X+0] = (\mathrm{id} \otimes \varepsilon),$$

where X and Y are arbitrary alphabets.

Coassociativity is clear since $f[X+(Y+Z)] = f[(X+Y)+Z]$.

Cocommutativity is clear since $f[X+Y] = f[Y+X]$.

On the power-sum basis the coproduct acts by sending $p_\lambda[X]$ to

$$\begin{aligned} p_\lambda[X+Y] &= \prod_{i \geqslant 1} p_{\lambda_i}[X+Y] = \prod_{i \geqslant 1} (p_{\lambda_i}[X] + p_{\lambda_i}[Y]) \\ &= \sum_{\substack{\nu^{(1)}, \nu^{(2)} \\ \nu^{(1)} \cup \nu^{(2)} = \lambda}} p_{\nu^{(1)}}[X]\, p_{\nu^{(2)}}[Y], \end{aligned}$$

from which cocommutativity follows. Alternatively, from the lectures,

$$\sum_\lambda \frac{p_\lambda[X]}{z_\lambda} = \sigma_1[X],$$

so that

$$\sum_\lambda \frac{p_\lambda[X+Y]}{z_\lambda} = \sigma_1[X+Y] = \sigma_1[X]\sigma_1[Y],$$

from which cocommutativity again follows.
For the counit,

$$p_\lambda[0] = \prod_i p_{\lambda_i}[0] = \delta_{l(\lambda),0} = \delta_{\lambda,0}$$

so that $\varepsilon \otimes \mathrm{id}$ forces $\nu^{(1)} = 0$. Hence $\nu^{(2)} = \lambda$. The second half follows by cocommutativity.
For coassociativity, note that the set of all triples of partitions $\nu^{(1)}, \nu^{(2,1)}, \nu^{(2,2)}$ such that $\nu^{(1)} \cup (\nu^{(2,1)} \cup \nu^{(2,2)}) = \lambda$ is equivalent to the set of all triples $\nu^{(1,1)}, \nu^{(1,2)}, \nu^{(2)}$ such that $(\nu^{(1,1)} \cup \nu^{(1,2)}) \cup \nu^{(2)} = \lambda$.
For $l(\lambda) = 1$, $\lambda = (r)$, say, the coproduct of p_r is $p_r[X+Y] = p_r[X] + p_r[Y]$, so that $\mu(p_r) = p_r \otimes 1 + 1 \otimes p_r$. The power sums p_r are thus primitive as per the above remark.

(c) Clearly

$$f[X]\, g[Y] \xmapsto{\mu} f[X]\, g[X] \xmapsto{\varepsilon} f[0]\, g[0] = (\varepsilon \otimes \varepsilon)(f[X]\, g[Y]).$$

Next, $\mu(1) = 1 \otimes 1$ as $\lambda = 0$ implies that $\nu^{(1)} = \nu^{(2)} = 0$. Alternatively, $1[X+Y] \equiv 1[X]\, 1[Y]$ since $p_0 = 1$ is the constant function.
We check the last axiom on the power-sum basis of Λ. Let us first consider primitive elements. The string diagram on the left gives the coproduct of $p_r\, p_s = p_{(r)\cup(s)}$. This matches the result of the diagram on the right:

$$\begin{aligned}
p_r \otimes p_s \xmapsto{\mu\otimes\mu}\ & (p_r \otimes 1 + 1 \otimes p_r) \otimes (p_s \otimes 1 + 1 \otimes p_s) \\
&= p_r \otimes 1 \otimes p_s \otimes 1 + p_r \otimes 1 \otimes 1 \otimes p_s + 1 \otimes p_r \otimes p_s \otimes 1 + 1 \otimes p_r \otimes 1 \otimes p_s \\
\xmapsto{\mathrm{id}\otimes\gamma\otimes\mathrm{id}}\ & p_r \otimes p_s \otimes 1 \otimes 1 + p_r \otimes 1 \otimes 1 \otimes p_s + 1 \otimes p_s \otimes p_r \otimes 1 + 1 \otimes 1 \otimes p_r \otimes p_s \\
\xmapsto{m\otimes m}\ & p_r\, p_s \otimes 1 + p_r \otimes p_s + p_s \otimes p_r + 1 \otimes p_r\, p_s \\
&= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} p_{\nu^{(1)}} \otimes p_{\nu^{(2)}}, \qquad\qquad \lambda := (r) \cup (s).
\end{aligned}$$

In general let $\lambda^{(1)}$ and $\lambda^{(2)}$ be two partitions. The string diagram on the left gives

$$p_{\lambda^{(1)}}[X]\, p_{\lambda^{(2)}}[Y] \xmapsto{m} p_{\lambda^{(1)}}[X]\, p_{\lambda^{(2)}}[X] = p_{\lambda^{(1)}\cup\lambda^{(2)}}[X] \xmapsto{\mu} p_{\lambda^{(1)}\cup\lambda^{(2)}}[X+Y].$$

This matches the result from the diagram on the right:

$$
\begin{aligned}
p_{\lambda^{(1)}}[X]\,p_{\lambda^{(2)}}[Y] \xmapsto{\mu\otimes\mu}\; & p_{\lambda^{(1)}}[X+X']\,p_{\lambda^{(2)}}[Y+Y'] \\
= & \sum_{\nu^{(i,1)}\cup\nu^{(i,2)}=\lambda^{(i)}} p_{\nu^{(1,1)}}[X]\,p_{\nu^{(1,2)}}[X']\,p_{\nu^{(2,1)}}[Y]\,p_{\nu^{(2,2)}}[Y'] \\
\xmapsto{\mathrm{id}\otimes\gamma\otimes\mathrm{id}} & \sum_{\nu^{(i,1)}\cup\nu^{(i,2)}=\lambda^{(i)}} p_{\nu^{(1,1)}}[X]\,p_{\nu^{(2,1)}}[X']\,p_{\nu^{(1,2)}}[Y]\,p_{\nu^{(2,2)}}[Y'] \\
\xmapsto{m\otimes m} & \sum_{\nu^{(i,1)}\cup\nu^{(i,2)}=\lambda^{(i)}} p_{\nu^{(1,1)}}[X]\,p_{\nu^{(2,1)}}[X]\,p_{\nu^{(1,2)}}[Y]\,p_{\nu^{(2,2)}}[Y] \\
= & \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} p_{\nu^{(1)}}[X]\,p_{\nu^{(2)}}[Y] \qquad \begin{aligned} \nu^{(i)} &:= \nu^{(1,i)}\cup\nu^{(2,i)} \\ \lambda &:= \lambda^{(1)}\cup\lambda^{(2)}, \end{aligned}
\end{aligned}
$$

where the final equality uses that the set of all subsets $\nu^{(i,1)}$ (which determines $\nu^{(i,2)}$) of λ_i (viewed as a list, not a diagram) for $i=1,2$ separately is the same as the set of all subsets $\nu^{(1)}$ of $\lambda=\lambda^{(1)}\cup\lambda^{(2)}$.

(d) By (co)commutativity it suffices to check the first equality:

$$f[X] \xmapsto{\mu} f[X+Y] \xmapsto{S\otimes\mathrm{id}} f[-X+Y] \xmapsto{m} f[-X+X] = f[0] = f[0]\,1[X].$$

Equivalently, on power sums,

$$S(p_\lambda) = S\Big(\prod_i p_{\lambda_i}\Big) = \prod_i S(p_{\lambda_i}) = (-1)^{l(\lambda)}\,p_\lambda,$$

where the ordering is irrelevant by commutativity. Thus

$$
\begin{aligned}
p_\lambda \xmapsto{\mu}\; & \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} p_{\nu^{(1)}}\otimes p_{\nu^{(2)}} \\
\xmapsto{S\otimes\mathrm{id}} & \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} (-1)^{l(\nu^{(1)})}\,p_{\nu^{(1)}}\otimes p_{\nu^{(2)}} \\
\xmapsto{m} & \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} (-1)^{l(\nu^{(1)})}\,p_\lambda = \delta_{\lambda,0}\,.
\end{aligned}
$$

Here the last equality uses that there are $\binom{l(\lambda)}{k}$ possible $\nu^{(1)}$ with $l(\nu^{(1)})=k$, whence

$$\sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} (-1)^{l(\nu^{(1)})} = \sum_{k=0}^{l(\lambda)} \binom{l(\lambda)}{k}(-1)^k = \big(1+(-1)\big)^{l(\lambda)} = \delta_{l(\lambda),0} = \delta_{\lambda,0}\,.$$

(e) Again compute with power sums: $\langle p_\lambda[0], 1\rangle = p_\lambda[0] = \delta_{\lambda,0} = \langle p_\lambda[X], 1[X]\rangle$, while

$$\begin{aligned}
\langle p_\lambda[X+Y], p_{\mu^{(1)}}[X]\, p_{\mu^{(2)}}[Y]\rangle &= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} \langle p_{\nu^{(1)}}[X]\, p_{\nu^{(2)}}[Y], p_{\mu^{(1)}}[X]\, p_{\mu^{(2)}}[Y]\rangle \\
&= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} \langle p_{\nu^{(1)}}[X], p_{\mu^{(1)}}[X]\rangle \langle\, p_{\nu^{(2)}}[Y], p_{\mu^{(2)}}[Y]\rangle \\
&= \sum_{\nu^{(1)}\cup\nu^{(2)}=\lambda} z^{-1}_{\mu^{(1)}}\, \delta_{\nu^{(1)},\,\mu^{(1)}}\, z^{-1}_{\mu^{(2)}}\, \delta_{\nu^{(2)},\,\mu^{(2)}} \\
&= z_\lambda^{-1}\, \delta_{\lambda,\mu} = \langle p_\lambda[X], p_\mu[X]\rangle, \qquad \mu := \mu^{(1)} \cup \mu^{(2)}.
\end{aligned}$$