Exercises

1. Dominance order. Recall that the dominance order ($\geqslant$) on the set $\{\lambda \vdash n\}$ of partitions of size n is defined by $\lambda \geqslant \mu$ if $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$ for all $i \geqslant 1$. This is a total order for $n \leqslant 5$ and a partial order for all $n \geqslant 6$. Show that $\lambda \geqslant \mu$ if and only if $\lambda' \leqslant \mu'$.

2. The centralizer of the symmetric group. Show that

$$z_\lambda := \prod_{i \geqslant 1} i^{m_i}\, m_i! = |Z_w|,$$

where $m_i = m_i(\lambda)$ is the multiplicity of i (the number of parts equal to i) in λ, $w \in S_n$ has cycle type λ (i.e., w has m_i cycles of length i), and Z_w is the centralizer of w.

3. Gaussian polynomials. Let n, m be nonnegative integers. Then the Gaussian polynomials or q-binomial coefficients are defined as

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\lambda \subseteq (m^n)} q^{|\lambda|}. \tag{1}$$

Here the sum is over all partitions λ that fit in a rectangle of height n and width m, i.e., partitions λ such that $\lambda_1 \leqslant m$ and $l(\lambda) \leqslant n$.

(a) Show that

(i) $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] = 1$ (initial condition);

(ii) $\left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} n+m \\ n \end{smallmatrix}\right]$ (symmetry);

(iii) $\left[\begin{smallmatrix} n+m \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right] + q^n \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right] = q^m \left[\begin{smallmatrix} n+m-1 \\ m \end{smallmatrix}\right] + \left[\begin{smallmatrix} n+m-1 \\ m-1 \end{smallmatrix}\right]$ (q-Pascal identities).

(b) Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{k=0}^{m} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}. \tag{2}$$

Remark. One similarly shows the q-Chu–Vandermonde or Durfee rectangle identity

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \sum_{\ell=0}^{n} q^{\ell(\ell+k)} \begin{bmatrix} m-k \\ \ell \end{bmatrix} \begin{bmatrix} n+k \\ \ell+k \end{bmatrix},$$

where k is an arbitrary integer, and $k = 0$ corresponds to the Durfee square identity. (The Durfee square of a partition is the largest square contained in its diagram.)

(c) Use (a) to construct the first six rows of the q-Pascal triangle and check that $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]$ computed this way matches the definition (1).

(d) Let $(a;q)_n := (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ denote a q-shifted factorial or q-Pochhammer symbol. Use (a) to show that

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^{n-m+1};q)_m}{(q;q)_m}.$$

(e) Let $[n] := (1-q^n)/(1-q) = 1+q+\cdots+q^{n-1}$ and $[n]! := [1][2]\dots[n]$. Show that

$$\begin{bmatrix} n+m \\ m \end{bmatrix} = \frac{[n+m]!}{[n]!\,[m]!} \quad \text{and} \quad \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}.$$

4. Plethystic notation. The aim of this question is to prove the q-binomial theorem

$$\sum_{k\geqslant 0} \frac{(a;q)_k}{(q;q)_k}\, z^k = \frac{(az;q)_\infty}{(z;q)_\infty} \tag{3}$$

using symmetric functions and plethystic notation. There are many alternative proofs, some of which are simpler, but hopefully this demonstrates the power of plethystic manipulations.

(a) To get a better feel for (3) show that, for n a nonnegative integer, it implies

(i) the q-binomial expansion

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} w^{n-k} z^k = \prod_{i=0}^{n-1} (w + q^i z);$$

(ii)

$$1 + \sum_{k\geqslant 1} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} z^k = \frac{1}{(z;q)_n}.$$

Remark. One can use (i) to prove Jacobi's triple product identity

$$\sum_{k=-\infty}^{\infty} (-z)^k q^{\binom{k}{2}} = (z;q)_\infty (q/z;q)_\infty (q;q)_\infty$$

by replacing $n \mapsto 2n$ followed by $k \mapsto k+n$ and using

$$(q^{-n}z;q)_{2n} = q^{-\binom{n+1}{2}} (-z)^n (z;q)_n (q/z;q)_n.$$

The triple product identity plays a key role in the theory of elliptic functions, and is the simplest example of a denominator identity for affine Kac–Moody Lie algebras, corresponding to $\mathrm{A}_1^{(1)}$ (affine $\mathfrak{sl}_2$).

(b) To prepare for the proof of (3) use plethystic notation to show that the generating function $\sigma_z[X] := \sum_{k\geqslant 0} h_k[X]\, z^k$ satisfies

(i) $\sigma_z[X+Y] = \sigma_z[X]\,\sigma_z[Y]$ and thus $h_k[X+Y] = \sum_{i=0}^{k} h_i[X]\, h_{k-i}[Y]$;

(ii) $\sigma_z[1] = \frac{1}{1-z}$ so $h_n[1] = 1$;

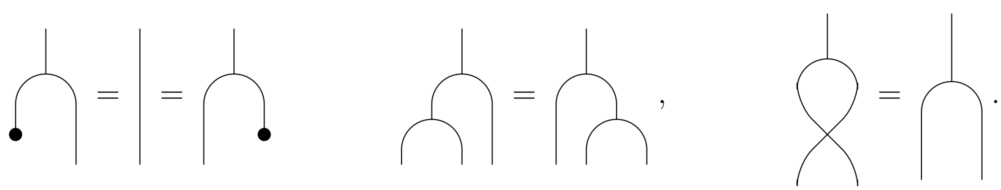
(iii) $\sigma_z[\frac{a}{1-q}] = \frac{1}{(az;q)_\infty}$.

(c) Now prove (3) by showing that both sides are equal to $\sigma_z[\frac{1-a}{1-q}]$.
Hint. For the left-hand side of (3) argue that it suffices to check equality with $\sigma_z[\frac{1-a}{1-q}]$ for a number of suitably chosen values of a and use part (b) to manipulate the alphabet on which h_r acts to recognise (2).

5. The Hopf-algebra structure of Λ. In this exercise we examine the algebraic structure of the ring of symmetric functions. All tensor products are over $K \coloneqq \mathbb{Z}$.

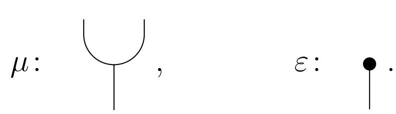
(a) Show that Λ is a commutative (unital associative) K-algebra with the usual product $m\colon \Lambda\otimes\Lambda \longrightarrow \Lambda$, $f[X]\,g[Y] \longmapsto f[X]\,g[X]$ in plethystic notation, and unit $e\colon K \longrightarrow \Lambda$ defined by $1 \longmapsto 1[X] \equiv 1$ extended K-linearly.
This structure can be nicely captured using string diagrams. Think of 'time' as increasing upwards, and depict (note that K is not drawn)

with $\gamma\colon \Lambda\otimes\Lambda \longrightarrow \Lambda\otimes\Lambda$ the permutation $f[X]\,g[Y] \longmapsto f[Y]\,g[X] = g[X]\,f[Y]$. The axioms of a (unital associative) algebra and commutativity then take the form

(4)


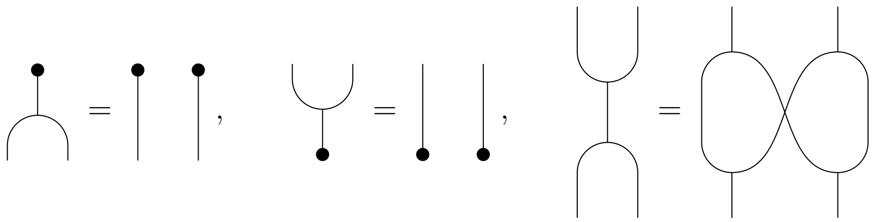
For example, the left-most diagram encodes $\Lambda \cong \Lambda\otimes K \xrightarrow{\mathrm{id}\otimes e} \Lambda\otimes\Lambda \xrightarrow{m} \Lambda$. Such an equality of string diagrams is often alternatively expressed as a commutative diagram.

(b) Show that Λ also is a cocommutative (counital coassociative) coalgebra, with coproduct $\mu\colon \Lambda \longrightarrow \Lambda\otimes\Lambda$ given by $f[X] \longmapsto f[X+Y]$ and counit $\varepsilon\colon \Lambda \longrightarrow K$, $f[X] \longmapsto f[0]$. Here the axioms are given by flipping all diagrams in (4) upside down and interpreting

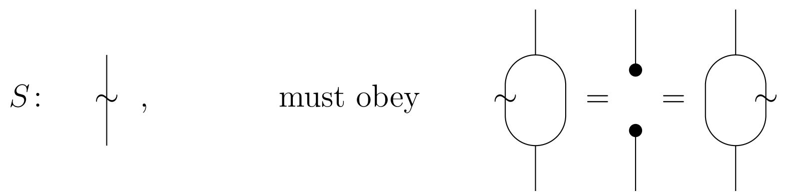


Do this using plethystic notation as well as by explicit computations for the power-sum basis p_λ.
Remark. Elements $f\in\Lambda$ whose coproduct satisfies $\mu(f) = f\otimes 1 + 1\otimes f$, such as the power sums p_r, are called primitive.

(c) Show that the preceding structures are compatible, making Λ a bialgebra: μ and ε are algebra homomorphisms (equivalently, m and e are coalgebra morphisms),



(d) Show that the bialgebra-structure of Λ extends to that of a Hopf algebra, with antipode $S\colon \Lambda \longrightarrow \Lambda$, $f[X] \longmapsto f[-X]$ extended as an (anti)homomorphism. Here the latter, depicted, say, as



Again do this using plethystic notation as well as by explicit calculation on the power-sum basis p_λ.
Remark. Viewed as equipped with this structure, Λ is commonly denoted by **Symm**.
Remark. Quantum integrability is related to quantum groups, which are sometimes defined as Hopf algebras that are quasitriangular, i.e., cocommutative up to conjugation by an invertible element of the tensor product of the Hopf algebra with itself, called the R-matrix, that behaves in a certain nice way under $\mu \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mu$ to guarantee it obeys the Yang–Baxter equation. Cocommutative Hopf algebras, such as Λ, are either viewed as boring examples ($R = 1 \otimes 1$) or excluded from the definition of a quantum group.

(e) Show that Λ is self dual with respect to the scalar product on $\Lambda \otimes \Lambda$ given by $\langle f_1[X]\, g_1[Y], f_2[X]\, g_2[Y] \rangle \coloneqq \langle f_1[X], f_2[X] \rangle \langle g_1[Y], g_2[Y] \rangle$ where the right-hand side features the Hall scalar product on Λ. That is, the algebra and coalgebra structure of Λ are dual in the sense that $\langle f[X+Y], g[X]\, h[Y] \rangle = \langle f[X], g[X]\, h[X] \rangle$ and $\langle f[0], n \rangle = \langle f[X], n\, 1[X] \rangle$, where the scalar product on K is given by multiplication.

6. Principal specialisation. Recall that the hook-length of a square $s = (i,j) \in \lambda$ is given by $h(s) = \lambda_i + \lambda_j - i - j + 1$. Show that

$$\prod_{h \in \mathscr{H}(\lambda)} (1 - q^h) = \prod_{i=1}^{n} (q^{n-i+1}; q)_{\lambda_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1 - q^{j-i}}{1 - q^{\lambda_i - \lambda_j + j - i}}, \tag{5}$$

where $\mathscr{H}(\lambda)$ denotes the multiset of hook-lengths of λ, n is any integer such that $n \geqslant l(\lambda)$ and $(a;q)_m = (1-a)(1-aq)\cdots(1-aq^{m-1})$. Use (5) to prove that

$$s_\lambda\left[\frac{1-a}{1-q}\right] = q^{n(\lambda)}\frac{\prod_{(i,j)\in\lambda}(1-aq^{j-i})}{\prod_{h\in\mathscr{H}(\lambda)}(1-q^h)} = q^{n(\lambda)}\frac{\prod_{i\geqslant 1}(aq^{1-i};q)_{\lambda_i}}{\prod_{h\in\mathscr{H}(\lambda)}(1-q^h)}, \tag{6}$$

where $n(\lambda) := \sum_{i\geqslant 1}(i-1)\lambda_i$. For $a = q^n$ and $l(\lambda) \leqslant n$ this is known as the principal specialisation formula for Schur functions.

7. Inverse branching rule. According to the branching rule for Schur functions,

$$s_\lambda[X+1] = \sum_{\mu\prec\lambda} s_\mu[X]. \tag{7}$$

Prove the combinatorial identity

$$\sum_{\substack{\mu'\prec\lambda'\\ \mu\succ\nu}} (-1)^{|\lambda/\mu|} = \delta_{\lambda\nu}, \tag{8}$$

and use this to show that

$$s_\lambda[X-1] = \sum_{\mu'\prec\lambda'} (-1)^{|\lambda/\mu|} s_\mu[X].$$

For example

$$s_{(3,1)}[X-1] = s_{(3,1)}[X] - s_{(3)}[X] - s_{(2,1)}[X] + s_{(2)}[X].$$

8. Kostant's multiplicity formula. Kostant's formula is an explicit (computationally not very efficient; Freudenthal's recursion formula is much more practical) expression for the weight multiplicities of irreducible representations of semi-simple Lie algebras, expressing the multiplicities as an alternating sum over what is known as the 'Kostant partition function'. In this question we look at a combinatorial analogue of this formula in the case of $\mathfrak{gl}_n$.
Recall that the Kostka number $K_{\lambda\alpha}$ counts the number of semistandard Young tableaux of shape λ and weight α, i.e., $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$.

(a) Show that $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$ and thus $h_\lambda = \sum_\mu P_{\lambda\mu} m_\mu$, where $P_{\lambda\mu} := \sum_\omega K_{\omega\lambda} K_{\omega\mu}$ will play the role of Kostant partition function.

(b) Using the RSK correspondence it may be shown that $P_{\alpha\beta}$ (for α and β (weak) compositions) counts the number of matrices with nonnegative integer entries such that the ith row-sum is β_i and the jth column-sum is α_j.
Count $P_{(2,1),(1,1,1)}$ by listing all pairs of semistandard tableaux contributing to the sum and by listing the relevant integer matrices.

(c) Use the Jacobi–Trudi formula to show that for μ a partition of length n,

$$\sum_{w\in S_n} \operatorname{sgn}(w) K_{\lambda, w(\mu+\delta)-\delta} = \delta_{\lambda\mu},$$

where $\delta := (n-1,\dots,1,0)$.

(d) For λ a partition of length n, prove the Kostant multiplicity formula

$$K_{\lambda\mu} = \sum_{w \in S_n} \operatorname{sgn}(w) P_{w(\lambda+\delta)-\delta,\mu}$$

and use it to compute $K_{(2,1),(1,1,1)}$.

9. Vertex operators. For n an integer define the linear operator $\alpha_n : \Lambda \to \Lambda$ by

$$\alpha_{-n} s_\mu = \sum_{\substack{\lambda \vdash |\mu|+n \\ \lambda/\mu = \text{border strip}}} (-1)^{\text{height}(\lambda/\mu)} s_\lambda \tag{9}$$

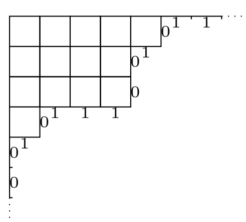
and

$$\alpha_n s_\lambda = \sum_{\substack{\mu \vdash |\lambda|-n \\ \lambda/\mu = \text{border strip}}} (-1)^{\text{height}(\lambda/\mu)} s_\mu$$

for $n \geqslant 0$.

(a) Show that α_n and α_{-n} are adjoint with respect to the Hall scalar product on Λ.

(b) Prove that the α_n satisfy the commutation relations of the Heisenberg algebra, i.e., $[\alpha_n, \alpha_m] = n\delta_{n,-m}$.
Hint. Use the representation of a partition in terms of its 0/1-sequence/edge sequence/code/Maya diagram. For example, the 0/1-sequence of the partition $(5,4,4,1)$ is



$\longleftrightarrow \quad \ldots 000101110010111 \ldots$

(c) Prove that the vertex operators

$$\Gamma_\pm(z) := \exp\left(\sum_{n\geqslant 1} \frac{z^n}{n}\,\alpha_{\pm n}\right)$$

obey the commutation relation

$$\Gamma_+(w)\Gamma_-(z) = \frac{1}{1-zw}\,\Gamma_-(z)\Gamma_+(w).$$

(d) Prove that

$$p_n s_\mu = \sum_{\substack{\lambda \vdash |\mu|+n \\ \lambda/\mu = \text{border strip}}} (-1)^{\text{height}(\lambda/\mu)} s_\lambda. \tag{10}$$

Remark. For $\lambda, \mu \vdash n$ let $\chi_\lambda(\mu)$ be the character of the irreducible representation of S_n indexed by λ evaluated at (elements of S_n in the conjugacy class indexed by) μ. From (d) and $p_\mu = \sum_\lambda \chi_\lambda(\mu) s_\lambda$ it follows that

$$\chi_\lambda(\mu) = \sum_{T \in \mathrm{BST}(\lambda,\mu)} (-1)^{\mathrm{height}(T)}.$$

where $\mathrm{BST}(\lambda, \mu)$ is the set of borderstrip tableaux of shape λ and weight μ, i.e., the set of tableaux of shape λ and weight μ such that the μ_i boxes filled with the letter i form a borderstrip, and where the height of a borderstrip tableau is the sum of the heights of the individual borderstrips making up the tableau. This is known as the Murnaghan–Nakayama rule.

(e) Use (d) to prove

(i) the 'Pieri rule'

$$\Gamma_-(z) s_\mu[X] = \sigma_z[X] s_\mu[X];$$

(ii) the 'branching rule'

$$\Gamma_+(z) s_\lambda[X] = s_\lambda[X + z];$$

(iii) the skew Schur function identity

$$s_{\lambda/\mu}(z_1, \ldots, z_n) = \big\langle \Gamma_+(z_1) \ldots \Gamma_+(z_n) s_\lambda, s_\mu \big\rangle.$$