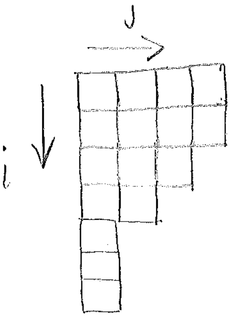
Symmetric functions

1 Partitions

$\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_i \in \mathbb{Z}$, $\lambda_i \geq \lambda_{i+1}$, $\lambda_i = 0$ for all $i > l(\lambda)$ for some $l(\lambda)$, the <u>length</u> of λ

$|\lambda| = \lambda_1 + \lambda_2 + \cdots$. If $|\lambda| = n$, $\lambda \vdash n$ (λ a partition of n)

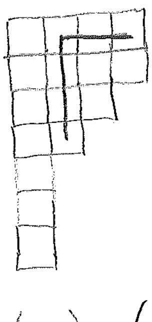
Eg $\lambda = (4,4,3,2,1,1,1,0,\dots) = (4,4,3,2,1,1,1)$, $l(\lambda) = 7$, $|\lambda| = 16$



<u>Young diagram</u> of λ (aka Ferrers diagram)

$s = (i,j) \in \lambda$ if $1 \leq i \leq l(\lambda)$, $1 \leq j \leq \lambda_i$

$h(s) = h(i,j) = \lambda_i + \lambda_j + j - i + 1$, the <u>hook length</u> of s.



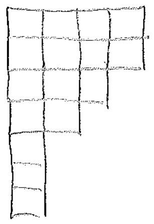
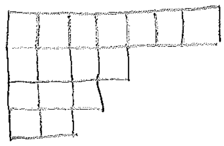
,

10	6	4	2
9	5	3	1
7	3	1	
5	1		
3			
2			
1			

$h(1,2) = 6$

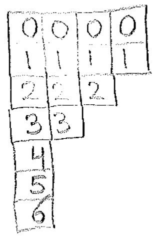
λ' the <u>conjugate</u> of λ

Eg $\lambda = (4,4,3,2,1,1,1)$, $\lambda' = (7,4,3,2)$

λ  , λ' 

$m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ <u>multiplicity</u> of parts of size i

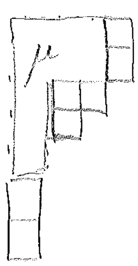
$$n(\lambda) = \sum_{i\geq 1} (i-1)\lambda_i = \sum_{i \geq 1} \binom{\lambda'_i}{2}$$



If $\lambda_i \geq \mu_i$ for all i, $\mu \subseteq \lambda$: μ is <u>contained</u> in λ

Skew shape/diagram: $\lambda - \mu$ (or λ/μ) for $\mu \subseteq \lambda$.

Eg $\lambda = (4,4,3,2,1,1,1)$, $\mu = (3,3,1,1,1)$

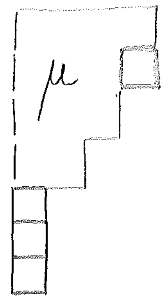
λ/μ 

If λ/μ has at most one square in each column:
<u>horizontal strip</u>

If λ/μ has at most one square in each row:
<u>vertical strip</u>

Eg $\lambda = (4,4,3,2,1,1,1)$, $\mu = (4,3,3,2)$

λ/μ is a vertical strip



If $\mu \subseteq \lambda$ and $(\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots)$ then λ & μ are said to be interlacing, denoted as $\lambda \succ \mu$

<u>Lemma</u> Let $\mu \subseteq \lambda$. Then $\lambda \succ \mu$ iff λ/μ is a horizontal strip.

Pf Note that for $(i,j) \in \lambda$ we have

$$(i,j) \in \lambda/\mu \iff \mu_i < j \leq \lambda_i .$$

Moreover if $(i,j) \in \lambda/\mu$ then also $(i,\lambda_i) \in \lambda/\mu$.

Now let $(i,j),(i+1,j)\in \lambda/\mu \Rightarrow \mu_i < j \leq \lambda_i$ & $\mu_{i+1} < j \leq \lambda_{i+1}$

(i.e $\begin{array}{c|c|} & j \\ \hline i & \\ \hline i+1 & \\ \hline \end{array} \in \lambda/\mu$)

This is incompatible with $\lambda_{i+1} \leq \mu_i$ since this would imply $\mu_{i+1} < j \leq \lambda_{i+1} \leq \mu_i < j \leq \lambda_i$, i.e., that $j<j$.

Hence $\lambda \succ \mu$ implies that λ/μ is a horizontal strip.

Conversely, if $(i+1,j)\in \lambda/\mu$ then $(i+1,\lambda_{i+1})\in\lambda/\mu$.

If λ/μ is a horizontal strip this implies $(i,\lambda_{i+1})\notin \lambda/\mu$

$\Rightarrow \mu_i \geq \lambda_{i+1} \Rightarrow \lambda \succ \mu.$

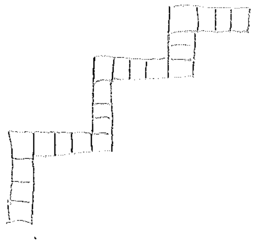
(by $\mu_i \geq j$ with $j=\lambda_{i+1}$) $\square$

Let $\lambda,\mu \vdash n$. Then the partial order defined by $\lambda \geq \mu$ if $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots \mu_i$ for all i, is called <u>dominance order</u>.

For all $n \leq 5$ dominance order is a total order, but not for any $n \geq 6$.

A skew diagram λ/μ is called a **border strip** (or ribbon) if λ/μ is connected and contains no 2×2 square $\boxplus$.

Eg



Eg.

$$
\begin{tikzcd}
& (6) \arrow[d, dash] & \\
& (5,1) \arrow[d, dash] & \\
& (4,2) \arrow[dl, dash] \arrow[dr, dash] & \\
(4,1^2) \arrow[dr, dash] & & (3^2) \arrow[dl, dash] \\
& (3,2,1) \arrow[dl, dash] \arrow[dr, dash] & \\
(3,1^3) \arrow[dr, dash] & & (2^3) \arrow[dl, dash] \\
& (2^2,1^2) \arrow[d, dash] & \\
& (2,1^4) \arrow[d, dash] & \\
& (1^6) &
\end{tikzcd}
$$

(2) The ring of symmetric functions

Let S_n be the symmetric group on n letters.

Then the ring of symmetric functions in $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}$ is defined as

$$\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{S_n}$$

Λ_n is graded by <u>degree</u>: $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$

where $\Lambda_n^k = \{ f \in \Lambda_n : \deg f = k \} \cup \{0\}$

Let $l(\lambda) \leq n$. Then the <u>monomial symmetric function</u> $m_\lambda(x_1, \dots, x_n)$ is defined as

$$m_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n / S_n^\lambda} w(x^\lambda),$$

where $x^\lambda := x_1^{\lambda_1} \dots x_n^{\lambda_n}$ and S_n^λ is the stabilizer of λ in S_n (ie $m_\lambda(x_1, \dots, x_n) = \sum_{\substack{\text{distinct} \\ \text{perm } \alpha \text{ of } \lambda}} x^\alpha$)

$\{m_\lambda\}_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}}$ is a $\mathbb{Z}$-basis of Λ_n^k

(Λ_n^k is a free $\mathbb{Z}$-module of rank $p(k,n)$, the # of partitions of k of length at most n)

The stability property

$$m_\lambda(x_1,\dots,x_{n-1},0) = \begin{cases} m_\lambda(x_1,\dots,x_{n-1}) & \text{if } \ell(\lambda) \leq n-1 \\ 0 & \text{if } \ell(\lambda) = n \end{cases}$$

may be used to define the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \dots$.

For $m \geq n$ define $\rho_{m,n} : \Lambda_m \to \Lambda_n$

$$m_\lambda(x_1,\dots,x_m) \mapsto \begin{cases} m_\lambda(x_1,\dots,x_n) & \ell(\lambda) \leq \\ 0 & \text{otherwise.} \end{cases}$$

$\rho_{m,n}$ is a surjective ring homomorphism with kernel

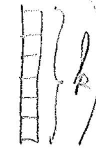
$$\ker \rho_{m,n} = \operatorname{Span}_{\mathbb{Z}} \{m_\lambda(x_1,\dots,x_m)\}_{n < \ell(\lambda) \leq m}$$

Note that for $m \geq l \geq n$, $\rho_{m,n} = \rho_{l,n} \circ \rho_{m,l}$ and that $\rho_{n,n}$ is the identity map on Λ_n.

This makes $\{(\Lambda_n)_{n \geq 0}, (\rho_{m,n})_{m \geq n \geq 0}\}$ an inverse system of $\mathbb{Z}$-modules.

Now define $\rho^k_{m,n} : \Lambda^k_m \to \Lambda^k_n$ by the restriction of $\rho_{m,n}$ to Λ^k_m ; $\rho^k_{m,n} = \rho_{m,n}\big|_{\Lambda^k_m}$

$\rho^k_{m,n}$ is injective, and hence bijective, if $m \geq n \geq k$

(The maximum length of a partition of size k is k: )

$\Lambda^k := \varprojlim_{n} \Lambda^k_n$, the inverse limit of the $\mathbb{Z}$-modules Λ^k_n relative to the homomorphisms $\rho^k_{m,n}$:

$f \in \Lambda^k$, $\quad f = (f_0, f_1, f_2, \dots)$

$f_n \in \Lambda^k_n$ for all n, such that $f_m(x_1, \dots, x_n, \underbrace{0, \dots, 0}_{m-n}) = f_n(x_1, \dots, x_n) \quad (m \geq n)$.

Hence $\rho_n^k : \Lambda^k \to \Lambda_n^k$ is an isomorphism for $n \geq k$.
$$f \mapsto f_n$$

This implies Λ^k is a free $\mathbb{Z}$-module with basis $\{m_\lambda\}_{\lambda \vdash k}$ where m_λ is defined by $\rho_n^k(m_\lambda) = m_\lambda(x_1,\dots,x_n)$, $n \geq k$.

$$\Lambda := \bigoplus_{k\geq 0} \Lambda^k$$

$\rho_n := \bigoplus_{k\geq 0} \rho_n^k : \Lambda \to \Lambda_n$ is an isomorphism for degrees $< n$

Elements of Λ are finite linear combinations of the m_λ, unlike the elements of $\hat{\Lambda} = \varprojlim_n \Lambda_n$ which allows for elements of unbounded degree.

Example:
$$m_0 = 1$$
$$m_1 = x_1 + x_2 + \cdots$$
$$m_{(2)} = x_1^2 + x_2^2 + \cdots$$
$$m_{(1^2)} = \sum_{1\leq i<j} x_i x_j$$
$$m_{(3)} = x_1^3 + x_2^3 + \cdots$$
$$m_{(2,1)} = \prod_{\substack{i,j\geq 1 \\ i\neq j}} x_i^2 x_j$$
$$m_{(1^3)} = \prod_{1\leq i<j<k} x_i x_j x_k$$

Besides the monomial symmetric functions there are several other important bases of Λ.

For $r \geq 0$ the rth <u>elementary symmetric function</u> e_r is defined as

$$e_r = m_{(1^r)} = \sum_{1 \leq i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Clearly
$$\sum_{r \geq 0} e_r z^r = \sum_{I \subseteq \mathbb{Z}} \prod_{i \in I} z x_i = \prod_{i \geq 1} \left(\sum_{k=0}^{1} (z x_i)^k \right)$$
$$= \prod_{i \geq 1} (1 + z x_i)$$

<u>Lemma</u>(*) $\Lambda = \mathbb{Z}[e_1, e_2, \dots]$ (and $\Lambda_n = \mathbb{Z}[e_1, \dots, e_n]$)

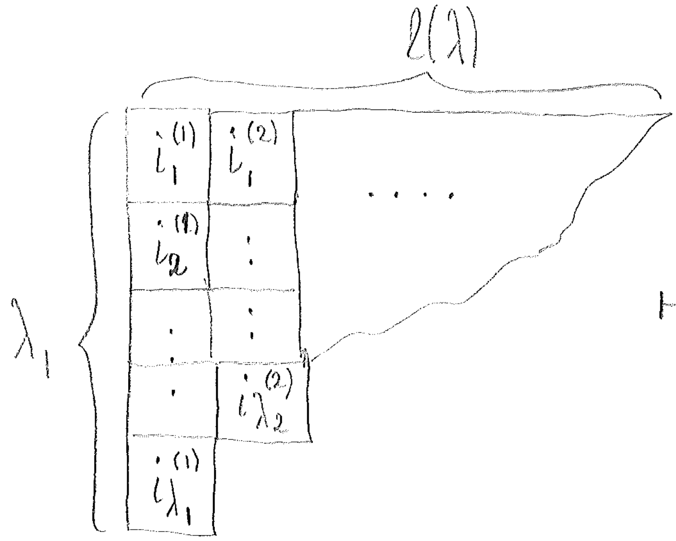
Pf Define $e_\lambda := \prod_{i \geq 1} e_{\lambda_i}$ (recall that $e_0 = 1$)

Each monomial contributing to e_r can be written as a column-strict tableau of shape (1^r):

$$\begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \\ \hline \\ \hline i_r \\ \hline \end{array} \longmapsto x_{i_1} x_{i_2} \cdots x_{i_r}$$

(*) fundamental theorem of symmetric functions.

Hence each monomial contributing to e_λ can be written as a column-strict tableau of shape λ':



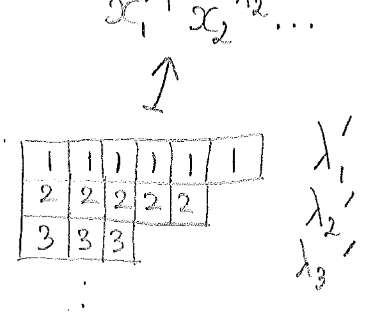
$$\mapsto \prod_{k\geq 1} x_{i_1^{(k)}} \cdots x_{i_{\lambda_k}^{(k)}}$$

Eg $\quad e_{(2,1,1)} = e_2 e_1^2$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} + \cdots + \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \text{etc.}$$

In mth row of of each tableau all entries must be greater or equal to m.

$\Rightarrow$ # m's is at most λ_m' :



$$\Rightarrow e_\lambda = m_{\lambda'} + \sum_{\mu < \lambda'} c_{\lambda\mu} m_\mu$$

$$\Rightarrow \{e_{\lambda'}\}_{\lambda \in P} \text{ forms a } \mathbb{Z}\text{-basis of } \Lambda \qquad \square$$

For $r \geq 0$ the rth <u>complete symmetric function</u> h_r is defined as

$$h_r = \sum_{\lambda \vdash r} m_\lambda = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Clearly, $$\sigma_z(x_1, x_2, \ldots) := \sum_{r \geq 0} h_r z^r = \prod_{i \geq 1} \left(\sum_{k \geq 0} (z x_i)^k \right)$$

$$= \prod_{i \geq 1} \frac{1}{1 - z x_i}$$

Comparing this with $\sum_{r \geq 0} e_r z^r = \prod_{i \geq 1} (1 + z x_i)$ shows that

$$\left(\sum_{r \geq 0} e_r (-z)^r \right) \sigma_z = 1$$

Hence $$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = \delta_{n,0} \qquad (*)$$

We shall see later using plethystic notation that e_r & h_r are really two sides of the same coin.

We can define an involution $\omega : \Lambda \to \Lambda$ by
$\omega(e_r) = h_r$.

That this indeed an involution follows from $(*)$:

$$\sum_{r=0}^{n} (-1)^r h_r\, \omega(h_{n-r}) = \delta_{n,0} \underset{r \to n-r}{\Rightarrow} \sum_{r=0}^{n} (-1)^r \omega(h_r)\, h_{n-r} = \delta_{n,0}$$

so that $\omega(h_r) = e_r$.

Consequently $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ & $\Lambda_n = \mathbb{Z}[h_1, \ldots, h_n]$

<u>Remark</u> In Λ_n, $e_r = 0$ for $r > n$ but $h_r \neq 0$ for $r > n$. However, the $h_1, h_2, \ldots$ are no longer algebraically independent. Eg, in Λ_2, $h_3 = 2h_2 h_1 - h_1^3 = 2h_{(2,1)} - h_{(1^3)}$

For $r \geq 1$ the rth <u>power sum</u> p_r is defined as

$$p_r = m_{(r)} = \sum_{i \geq 1} x_i^r$$

Then $\psi_z(x_1, x_2, \ldots) := \sum_{r \geq 1} \frac{p_r z^r}{r} = $ p.t.o.

$$= \sum_{r\geq 1} \sum_{i\geq 1} \frac{(z x_i)^r}{r} = -\sum_{i\geq 1} \log(1 - z x_i)$$

$$= \log \sigma_z$$

In other words, $\sigma_z = e^{\psi_z}$ and $\sigma_z' = \psi_z' \sigma_z$

This implies <u>Newton's relations</u>

$$n h_n = \sum_{r=1}^{n} p_r h_{n-r} \qquad (*)$$

Pf. $\times$ by z^{n-1} & $\sum_{n\geq 1}$ $\Rightarrow$ $\sigma_z' = \sum_{r\geq 1} \sum_{n\geq r} p_r h_{n-r} z^{n-1}$

$$= \sum_{r\geq 1} p_r z^{r-1} \sum_{n\geq 0} h_n z^n = \psi_z' \sigma_z . \qquad \square$$

An immediate consequence of $(*)$ is that

$$\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$$

but the power sums do not form a $\mathbb{Z}$-basis of Λ:

$$h_2 = \frac{1}{2}(p_1^2 + p_2)$$

Set $p_0 := 1$ and define $p_\lambda := \prod_{i \geq 1} p_{\lambda_i}$ &

$z_\lambda := \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$ (see tutorial question 2)

<u>Lemma</u> $\sigma_z = \sum_\lambda \frac{p_\lambda z^{|\lambda|}}{z_\lambda}$ i.e., $h_r = \sum_{\lambda \vdash r} \frac{p_\lambda}{z_\lambda}$

<u>Pf</u> $$\sigma_z = e^{\psi_z} = e^{\sum_{r \geq 1} \frac{p_r z^r}{r}} = \prod_{r \geq 1} e^{\frac{p_r z^r}{r}}$$

$$= \prod_{r \geq 1} \sum_{m_r} \frac{\left(\frac{p_r z^r}{r}\right)^{m_r}}{m_r!} = \sum_\lambda \frac{p_\lambda z^{|\lambda|}}{z_\lambda}$$

$\lambda := (1^{m_1} 2^{m_2} \ldots)$
so that $|\lambda| = \sum_{r \geq 1} r m_r$ $\square$

(3) Plethystic or λ-ring notation

The ring Λ may be viewed as a free λ-ring in a single variable. Without formally defining λ-rings we briefly discuss some convenient notation stemming from this point of view.

Since $f \in \Lambda$ is symmetric it is natural to think of symmetric functions as operators acting on sets, like $\{x_1, x_2, \ldots\}$, which we will call alphabets. Instead of the usual set notation, we adopt additive notation, writing

$$X = \{x_1, x_2, \ldots\} = x_1 + x_2 + \cdots$$

To avoid confusion, when such notation is used for symmetric functions, plethystic brackets $[\cdot]$ are used; $f(X) = f(x_1, x_2, \ldots) = f[X] = f[x_1 + x_2 + \cdots]$

The idea is now to allow for more complicated alphabets, not all of which are necessarily countable. (When an alphabet X is countable, we can write $X = \sum_{x \in X} x$).

First we simply consider $X + Y := \sum_{x \in X} x + \sum_{y \in Y} y$ for countable alphabets (set union if X & Y are disjoint). Obviously, by the definition of p_r,

$$p_r[X+Y] = p_r[X] + p_r[Y],$$

and for example $p_r[\underbrace{X + \dots + X}_{n \text{ times}}] = p_r[nX] = n p_r[X]$

For arbitrary alphabets (we are yet to construct example of non-countable alphabets) we use the same definition of $X+Y$ (or p_r acting on $X+Y$)

$$p_r[X+Y] := p_r[X] + p_r[Y].$$

Given X, Y we now form $X - Y$ as the alphabet such that

$$p_r[X \overset{\text{plethystic-sign}}{-} Y] = p_r[X] \overset{\text{normal-sign}}{-} p_r[Y], \quad r \geq 1$$

as well as XY $\left(= \left(\sum_{x \in X} x\right)\left(\sum_{y \in Y} y\right)\right.$ in the countable case$\left.\right)$ i.e. Cartesian product

$$p_r[XY] = p_r[X]\, p_r[Y].$$

Note that we can manipulate alphabets as if they are ordinary elements of a commutative ring

$$p_r[(X-Y)+Y] = p_r[X+(Y-Y)] = p_r[X]$$

$$p_r[X(Y-Z)] = p_r[XY - XZ] \quad \text{etc.}$$

We have addition & multiplication but only a special case of division

$$p_r\left[\frac{X}{1-q}\right] := \frac{p_r[X]}{1-q^r} = p_r[X]\, p_r[1+q+q^2+\dots]$$

$$= p_r[X(1+q+q^2+\dots)] \qquad r \geq 1$$

Note that $p_r\left[\frac{X(1-q)}{1-q}\right] \overset{①}{=} p_r[X]$

$$\overset{②}{=} p_r\left[\frac{X}{1-q}\right] p_r[1-q]$$

$$= \frac{p_r[X]}{1-q^r}(1-q^r) = p_r[X]$$

(In other words $1-q$ & $1+q+q^2+\dots$ are units).

Letters in an alphabet should not be confused with ordinary "scalars" or what are sometimes referred to as <u>binomial variables</u>. For example if z is a single letter alphabet then

$$p_r[zX] = z^r p_r[X]. \quad \text{But } p_r[\underset{\text{binomial variable}}{n}X] = n\, p_r[X]$$

(More generally, for ξ a binomial variable (eg $\xi \in \mathbb{R}$)

$$p_r[\xi X] := \xi\, p_r[X]).$$

Sometimes it is also convenient to use an ordinary minus sign in plethystic notation, so we can represent the set of variables $\{-x_1, -x_2, \dots\} =: \varepsilon X$

Then $p_r[\varepsilon X] = (-1)^r p_r[X]$, so that

$$p_r[-\varepsilon X] = (-1)^{r-1} p_r[X] \qquad (*)$$

__Lemma__ • $e_r[X] = (-1)^r h_r[-X]$

• $\omega : \Lambda \to \Lambda$ corresponds to the __plethystic substitution__ $X \mapsto -\varepsilon X$.

__Remark__ More generally, plethystic substitutions correspond to ring homomorphism, e.g.,

$$\begin{cases} \Lambda \to \Lambda \\ X \mapsto -\varepsilon X \end{cases}, \quad \begin{matrix} \Lambda \to \Lambda \otimes \Lambda \\ X \to X \pm Y \end{matrix}, \quad \begin{matrix} \Lambda \to \Lambda \otimes \mathbb{Q}(q) \\ X \to \dfrac{X}{1-q} \end{matrix}$$

__Pf__ • $\sigma_z[-X] = e^{\Psi_z[-X]} \underset{\underset{p_r[-X] = -p_r[X]}{\uparrow}}{=} e^{-\Psi_z[X]} = \dfrac{1}{\sigma_z[X]}$

$$= \sum_{r \geq 0} e_r[X](-z)^r \Rightarrow h_r[-X] = (-1)^r e_r[X]$$

• $\omega(\sigma_z[X]) = \sum_{r \geq 0} e_r[X] z^r = \sum_{r \geq 0} h_r[-X](-z)^r = \sigma_{-z}[-X]$

Hence, since $\psi_z = \log \sigma_z$,

$$\omega(\psi_z[X]) = \psi_{-z}[-X], \text{ i.e.,}$$

$$\omega(p_r[X]) = (-1)^r p_r[-X] = p_r[-\varepsilon X] \qquad \square$$

<u>Corollary</u> $\omega(p_r) = (-1)^{r-1} p_r$

Pf This immediately follows from (*).

<u>Remark</u> Plethystic substitutions are closely related to the structure of Λ as a self-dual, cocommutative, graded <u>Hopf algebra</u>. The <u>comultiplication</u> $\mu : \Lambda \to \Lambda \otimes \Lambda$ can be realised as $f[X] \mapsto f[X+Y]$, the <u>multiplication</u> $m : \Lambda \otimes \Lambda \to \Lambda$ as $f[X]g[Y] \mapsto f[X]g[X]$ and the <u>antipode</u> $S : \Lambda \to \Lambda$ as $f[X] \mapsto f[-X]$.

We will explore this more in exercise 5.