4 The Hall scalar product

Definition The Cauchy product of two alphabets X & Y is given by $\sigma_1[XY]$

Lemma
$$\sigma_1[XY] = \sum_\lambda \frac{1}{z_\lambda} p_\lambda[X]\, p_\lambda[Y] \qquad (1)$$
$$= \sum_\lambda h_\lambda[X]\, m_\lambda[Y] \qquad (2)$$

Pf
$$\sigma_1[XY] = \sum_\lambda \frac{p_\lambda[XY]}{z_\lambda} \underset{\substack{\uparrow \\ p_r[XY]=p_r[X]p_r[Y]}}{=} \sum_\lambda \frac{p_\lambda[X]\, p_\lambda[Y]}{z_\lambda}$$

which gives (1). For (2) it suffices to consider $Y = y_1 + \cdots + y_n$. Then

$$\sigma_1[XY] \underset{\substack{\uparrow \\ XY=\sum_i X y_i \ \&\ \sigma_z[A+B]=\sigma_z[A]\sigma_z[B]}}{=} \prod_{i=1}^{n} \sigma_{y_i}[X] = \prod_{i=1}^{n} \left(\sum_{r_i \geq 0} h_{r_i}[X]\, y_i^{r_i} \right)$$

$$\underset{\substack{\uparrow \\ \alpha=(r_1,\ldots,r_n) \\ \text{composition}}}{=} \sum_\alpha \left(\prod_{i=1}^{n} h_{\alpha_i}[X] \right) y^\alpha \qquad (y^\alpha := y_1^{\alpha_1} \cdots y_n^{\alpha_n})$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} h_\lambda[X] \sum_{w\in S_n/S_n^\lambda} w(y^\lambda)$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} h_\lambda[X]\, m_\lambda[Y] \qquad \square$$

Definition (Hall scalar product on Λ)

$\langle\ ,\ \rangle : \Lambda\times\Lambda \to \mathbb{Z}$

$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$

Proposition Let (a_λ) & (b_λ) be bases of Λ. Then

$$\langle a_\lambda, b_\mu\rangle = \delta_{\lambda\mu} \quad (1) \quad \text{iff} \quad \sum_\lambda a_\lambda[X]\, b_\lambda[X] = \sigma_1[XY]$$

Remark
- (a_λ) & (b_λ) as above are referred to as dual bases w.r.t. the Hall scalar product
- The lin. operator $f^\perp : \Lambda\to\Lambda$ for $f\in\Lambda$ is defined as $\langle f^\perp g, h\rangle = \langle g, fh\rangle$ and referred to as the adjoint of multiplication by f.

Pf We have $a_\lambda = \sum_\nu c_{\lambda\nu} h_\nu$, $b_\mu = \sum_\omega d_{\omega\mu} m_\omega$ so that

$\langle a_\lambda, b_\mu \rangle = \sum_\nu c_{\lambda\nu} d_{\nu\mu}$. Hence (1) is equivalent to

$$\sum_\nu c_{\lambda\nu} d_{\nu\mu} = \delta_{\lambda\mu} \Leftrightarrow \sum_\lambda d_{\mu\lambda} c_{\lambda\nu} = \delta_{\mu\nu}$$

Also

$$\sum_\lambda a_\lambda[X] b_\lambda[Y] \overset{(1)}{=} \sum_{\lambda,\mu,\nu} d_{\mu\lambda} c_{\lambda\nu} h_\nu[X] m_\mu[Y]$$

$$\overset{(2)}{=} \sigma_1[XY]$$

$$= \sum_\mu h_\mu[X] m_\mu[Y]$$

Hence (2) is also equivalent to $\sum_\lambda d_{\mu\lambda} c_{\lambda\nu} = \delta_{\mu\nu}$ $\square$

5 Schur functions

Let $\ell(\lambda) \leq n$. Then the Schur function $s_\lambda(x_1, \dots, x_n)$ is defined as

$$s_\lambda(x_1,\dots,x_n) := \frac{\det\limits_{1\leq i,j\leq n}\left(x_i^{\lambda_j+n-j}\right)}{\prod\limits_{1\leq i<j\leq n}(x_i-x_j)} = \frac{\sum\limits_{w\in S_n} \operatorname{sgn}(w)\, w\left(x^{\lambda+\delta}\right)}{\prod\limits_{i<j}(x_i-x_j)}$$

where $\delta = (n-1, \dots, 2, 1, 0)$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and

$$\prod_{1\leq i<j\leq n}(x_i-x_j) = \det_{1\leq i,j\leq n}\left(x_i^{n-j}\right) \quad \left(= \Delta(x_1,\dots,x_n)\right)$$

are the Vandermonde product and determinant respectively.

Lemma The Schur function s_λ is a symmetric polynomial of degree $|\lambda|$ and $\{s_\lambda(x_1,\dots,x_n)\}_{\substack{\ell(\lambda)\leq n\\ \lambda\vdash k}}$ forms a $\mathbb{Z}$-basis of Λ_n^k.

Pf Symmetry is obvious since both the numerator and denominator are skew symmetric polynomials. Since the numerator vanishes if $x_i = x_j$ for some $1 \leq i < j \leq n$ polynomiality is also clear.

The degree claim is also obvious.

Now, since $\left\{ \det_{1 \leq i,j \leq n} \left(x_i^{\mu_j} \right) \right\}_{\substack{\ell(\mu) \leq n \\ \mu \text{ strict}}}$ is a basis of the free $\mathbb{Z}$-module A_n of skew-symmetric polynomials in $x_1, \dots, x_n$ (We are anti-symmetrising x^μ which vanishes unless μ is strict). But a strict partition μ of length at most n can be represented as $\mu = \delta + \lambda$. Since
$$\varphi : \Lambda_n \to A_n$$
$$f \mapsto \Delta f$$
is an isomorphism, $\{ s_\lambda \}_{\ell(\lambda) \leq n}$ is a $\mathbb{Z}$-basis for Λ_n

$\square$

Remark The same determinant definition may be used to define the Schur functions for arbitrary (weak) compositions $\alpha = (\alpha_1, \dots, \alpha_n)$. Then, if $\alpha = w(\lambda + \delta) - \delta$ for some partition λ and $w \in S_n$, then $s_\alpha = \mathrm{sgn}(w) s_\lambda$. Otherwise $s_\alpha = 0$

Lemma The Schur functions are stable:

$$s_\lambda(x_1,\dots,x_{n-1},0) = \begin{cases} s_\lambda(x_1,\dots,x_{n-1}) & \text{if } \ell(\lambda)\le n-1 \\ 0 & \text{otherwise} \end{cases}$$

Pf

$$s_\lambda(x_1,\dots,x_n) \overset{\lambda_n=0}{\underset{①}{=}} \frac{\det\limits_{1\le i,j\le n-1}\left(x_i^{\lambda_j+n-j}\right)}{\prod\limits_{1\le i<j\le n-1}(x_i-x_j)\prod\limits_{i=1}^{n} x_i}$$

$$= \frac{\det\limits_{1\le i,j\le n-1}\left(x_i^{\lambda_j+(n-1)-j}\right)}{\prod\limits_{1\le i,j\le n-1}(x_i-x_j)}$$

$$\overset{\lambda_n>0}{\underset{②}{=}} 0 \qquad \square$$

It thus makes sense to define $s_\lambda(x_1,\dots,x_n)=0$ if $\ell(\lambda)>n$. Moreover, $s_\lambda(x_1,x_2,\dots)$ is well-defined and $\{s_\lambda\}$ forms a $\mathbb{Z}$-basis of Λ.

<u>Remark</u> It may be shown that the Schur functions on n letters are the characters of the polynomial representations of $GL(n,\mathbb{C})$ as well as related to the characters of S_n, see exercise 9. The occurence of both $GL(n,\mathbb{C})$ & S_n can be understood through Schur-Weyl duality.

A semistandard Young tableau of shape λ and content/filling/weight α on n letters is a filling of the Young diagram of λ with the numbers $1,2,..,n$ such that row are weakly increasing from left to right and strictly increasing from top to bottom, and such that there are α_i boxes filled with i. (Hence $|\lambda|=|\alpha|$)

E.g

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 4 & 6 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \quad , \quad \begin{array}{l}\lambda = (5,4,2,2,1) \\ \\ \alpha = (2,2,1,4,2,3)\end{array}$$

Note that a Young tableau T of shape λ can alternatively be represented by a sequence of interlacing partitions:

$$0 = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(n)} = \lambda$$

where the skew shape $\lambda^{(i)}/\lambda^{(i-1)}$ represents those boxes of T filled with i, i.e., $|\lambda^{(i)}/\lambda^{(i-1)}| = \alpha_i$

Eg

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 4 & 6 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \qquad 0 \prec (2) \prec (3,1) \prec (3,2) \prec (5,3,1) \prec (5,3,2,1) \prec (5,4,2,2,1)$$

Let $SSYT(\lambda, \alpha) = \{ T : \text{shape}(T) = \lambda, \text{content}(T) = \alpha \}$

<u>Theorem</u> $$s_\lambda = \sum_T x^T = \sum_{T \in SSYT(\lambda, \cdot)} x^{\text{content}(T)}$$

where, if $x = (x_1, \dots, x_n)$, all T are on n letters.

Eg $s_{(2,1)}(x_1,x_2,x_3) = m_{(2,1)}(x_1,x_2,x_3) + 2\, m_{(1^3)}(x_1,x_2,x_3)$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$x_1^2 x_2 \quad x_1^2 x_3 \quad x_1 x_2^2 \quad x_2^2 x_3 \quad x_1 x_3^2 \quad x_2 x_3^2 \quad \underbrace{\qquad\qquad}_{2 x_1 x_2 x_3}$$

<u>Remark</u> Since s_λ is symmetric, the theorem implies $|SSYT(\lambda,\alpha)| = |SSYT(\lambda, w(\alpha))|$, $w \in S_n$.

This number is known as the Koska number $K_{\lambda\alpha}$.

More generally, in the representation theory of semi-simple Lie algebras

$$\mathrm{char}_{V(\lambda)} = \sum_{\mu \in \mathfrak{h}^*} \mathrm{mult}(\mu)\, e^{\mu} = \sum_{\mu \in \mathfrak{h}^*} K_{\lambda\mu}\, e^{\mu}$$

where $V(\lambda)$ is an irreducible $\mathfrak{g}$-module of highest weight λ, μ is an arbitrary weight and $K_{\lambda\mu} = \mathrm{mult}(\mu)$ is the dimension of the weight space indexed by μ in the weight space decomposition of $V(\lambda)$.

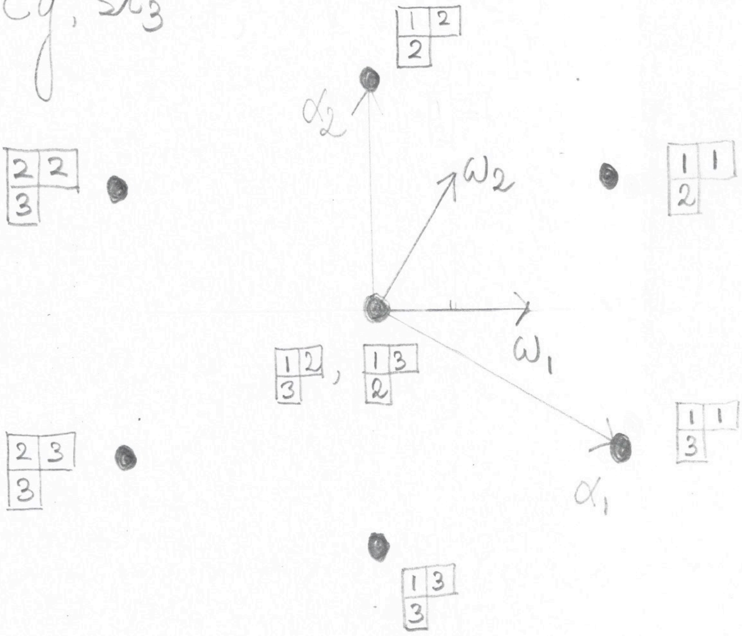
We also note that, since $K_{\lambda\alpha} = K_{\lambda w(\alpha)}$ we may write

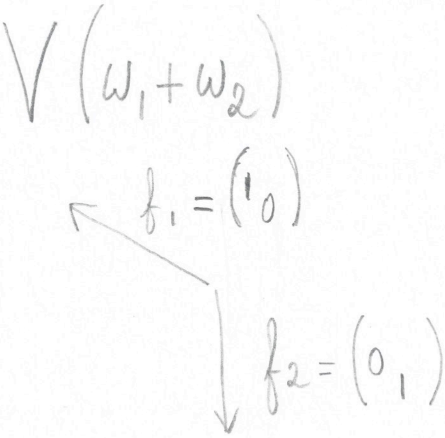
$$s_\lambda = \sum_{\mu} K_{\lambda\mu}\, m_\mu$$

summed over partitions μ such that $|\mu| = |\lambda|$.

(30')

Eg. sl_3





$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} = (e_1\wedge e_2)\otimes e_1 \quad ; \quad f_2\,\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} = f_2(e_1\wedge e_2)\otimes e_1 \oplus (e_1\wedge e_2)\otimes \underbrace{f_2 e_1}_{=0}$$

$$\begin{pmatrix}0\\1\end{pmatrix}$$

$$= (e_1\wedge e_3)\otimes e_1$$

$$= \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}$$

$$\begin{pmatrix}1\\0\end{pmatrix}$$

$$\| $$

$$f_1\,\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} = \overbrace{f_1(e_1\wedge e_2)}^{0}\otimes e_1 \oplus (e_1\wedge e_2)\otimes f_1 e_1$$

$$= (e_1\wedge e_2)\otimes e_2 = \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$$

$$f_2\,\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} = f_2(e_1\wedge e_2)\otimes e_2 \oplus (e_1\wedge e_2)\otimes f_2 e_2 = (e_1\wedge e_3)\otimes e_2 \oplus (e_1\wedge e_2)\otimes e_3$$

$$= \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

Pf By the correspondence between semi-standard tableaux and sequences of interlacing partitions,

$$\sum_T x^T = \sum_{T \in SSYT(\lambda,\cdot)} x^{\mathrm{content}(T)}$$

$$= \sum_{0=\lambda^{(0)} \prec \cdots \prec \lambda^{(n)} = \lambda} \prod_{i=1}^{n} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|}$$

$$\underset{\substack{\uparrow \\ \mu := \lambda^{(n-1)}}}{=} \sum_{\mu \prec \lambda} x_n^{|\lambda/\mu|} \sum_{0=\lambda^{(0)} \prec \cdots \prec \lambda^{(n-1)} = \mu} \prod_{i=1}^{n-1} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|}$$

$$= \sum_{\mu \prec \lambda} x_n^{|\lambda/\mu|} S_\mu(x_1,\ldots,x_{n-1})$$

It thus suffices to prove the <u>branching rule</u>

$$S_\lambda(x_1,\ldots,x_n) = \sum_{\mu \prec \lambda} x_n^{|\lambda/\mu|} S_\mu(x_1,\ldots,x_{n-1})$$

(Clearly both descriptions of the Schur functions satisfy the same initial condition $S_\lambda[\underset{\substack{\uparrow \\ \text{empty alphabet}}}{\underline{0}}] = \delta_{\lambda 0}$

By homogeneity it suffices to prove

$$s_\lambda(x_1, \dots, x_{n-1}, 1) = \sum_{\mu \prec \lambda} s_\mu(x_1, \dots, x_{n-1})$$

$$\left(\text{so that } s_\lambda[X+1] = \sum_{\mu \prec \lambda} s_\mu[X] \right)$$

From the determinantal definition of s_λ:

$$s_\lambda(x_1, \dots, x_{n-1}, 1) = \frac{\det \begin{pmatrix} x_i^{\lambda_j + n - j} & i < n \\ 1 & i = n \end{pmatrix}}{\Delta(x_1, \dots, x_{n-1}) \prod_{i=1}^{n-1} (x_i - 1)}$$

subtract last row from row i & divide by $(x_i - 1)$ $\longrightarrow$ $$= \det \begin{pmatrix} \sum_{k=0}^{\lambda_j + n - j - 1} x_i^k & i < n \\ 1 & i = n \end{pmatrix} \Big/ \Delta_{n-1}$$

subtract column 2 from col 1

" C3 from C2

etc $\longrightarrow$ $$= \det \begin{pmatrix} \sum_{k = \lambda_{j+1} + n - j - 1}^{\lambda_j + n - j - 1} x_i^k & i < n \\ \delta_{jn} & i = n \end{pmatrix} \Big/ \Delta_{n-1}$$

$$= \det_{1 \leq i, j \leq n-1} \left(\sum_{\mu_j = \lambda_{j+1}}^{\lambda_j} x_i^{\mu_j + n - j - 1} \right) \Big/ \Delta_{n-1}$$

multilinearity $\longrightarrow$ $$= \sum_{\mu \prec \lambda} \underbrace{\det_{1 \leq i, j \leq n-1} \left(x_i^{\mu_j + (n-1) - j} \right) \Big/ \Delta_{n-1}}_{s_\mu(x_1, \dots, x_{n-1})}$$

$\square$

The branching rule can also be written as

$$s_\lambda(x_1,\dots,x_n) = \sum_\mu s_{\lambda/\mu}(x_n)\, s_\mu(x_1,\dots,x_{n-1})$$

where the <u>skew Schur function</u> is defined by

$$s_\lambda[X+Y] = \sum_\mu s_{\lambda/\mu}[X]\, s_\mu[Y]$$

Clearly, $s_{\lambda/\mu}(z) = \begin{cases} z^{|\lambda/\mu|} & \text{if } \lambda \succ \mu \\ 0 & \text{otherwise} \end{cases}$

and, more generally, for $X = x_1 + x_2 + \cdots + x_n$

$$s_{\lambda/\mu}[X] = \prod_{i=1}^{n} s_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i) \quad , \ \lambda^{(n)} := \lambda, \ \lambda^{(0)} := \mu$$

Hence also

$$s_{\lambda/\mu} = \sum_{SSYT(\lambda/\mu,\,\cdot)} x^T$$

<u>Remark</u> We have

$$
\begin{aligned}
s_\lambda[X+u+v] &\overset{①}{=} \sum_\mu s_{\lambda/\mu}(u)\, s_\mu[X+v] \\
&= \sum_{\mu,\nu} s_{\lambda/\mu}(u)\, s_{\mu/\nu}(v)\, s_\nu[X] \\
&\overset{②}{=} \sum_\mu s_{\lambda/\mu}(v)\, s_\mu[X+u] \\
&= \sum_{\mu,\nu} s_{\lambda/\mu}(v)\, s_{\mu/\nu}(u)\, s_\nu[X]
\end{aligned}
$$

and thus $\displaystyle\sum_\mu s_{\lambda/\mu}(u)\, s_{\mu/\nu}(v) = \sum_\mu s_{\lambda/\mu}(v)\, s_{\mu/\nu}(u)$.

Defining the 'transfer matrix' $T(u)$ with $\big(T(u)\big)_{\lambda,\mu} = s_{\lambda/\mu}(u)$, we see that

$T(u)\,T(v) = T(v)\,T(u)$, a hallmark of quantum integrability.

Theorem (Jacobi-Trudi identity)

Let λ be a partition of length at most k. Then

$$s_\lambda = \det_{1\le i,j\le k}\left(h_{\lambda_i+j-i}\right) \qquad (h_r := 0 \text{ for } r<0)$$

Remark • More generally $s_{\lambda/\mu} = \det_{1\le i,j\le k}\left(h_{\lambda_i-\mu_j+j-i}\right)$

Proof We sketch the proof, which is essentially an application of a special case of the Lindström-Gessel-Viennot lemma.

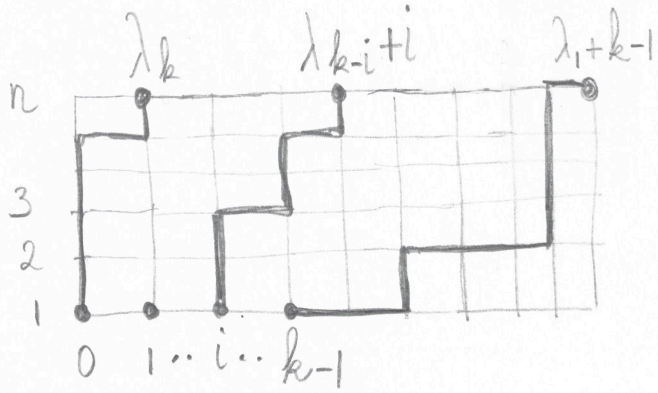
Wlog we may assume $\ell(\lambda)=k$.

Recall $s_\lambda = \sum_{T\in SSYT(\lambda,\cdot)} x^{\text{content}(T)}$

Each T in SSYT is in bijection with a set of nonintersecting lattice paths in a rectangular grid as follows.

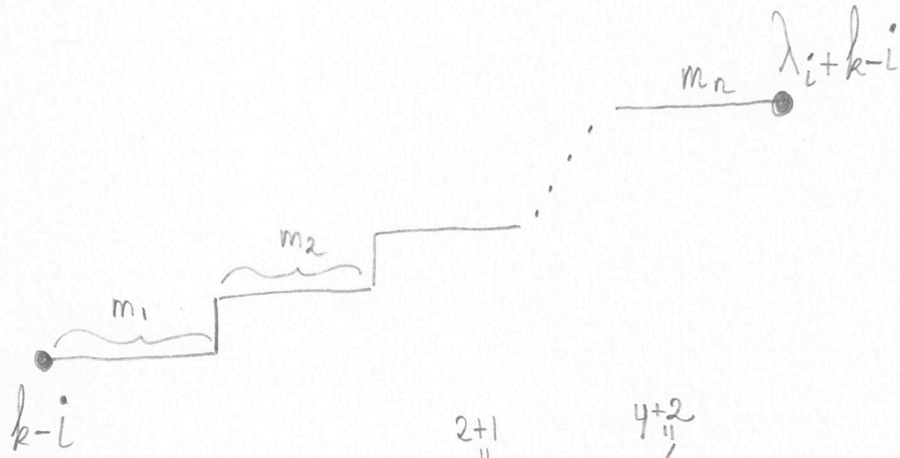
If $T \in SSYT(\lambda, \alpha)$, $\lambda = (\lambda_1, \dots, \lambda_k)$

$\alpha = (\alpha_1, \dots, \alpha_n)$



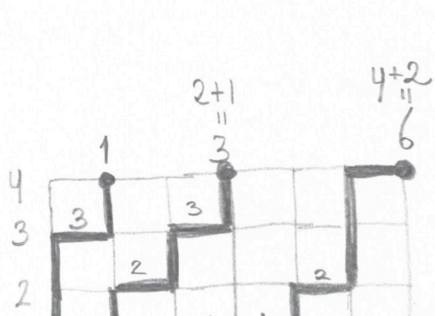
$\longleftrightarrow \quad T$

where, if the ith row of T has entries $1^{m_1} 2^{m_2} \dots n^{m_n}$ then the ith path from the right is



E.g. $T =$

1	1	2	4
2	3		
3			

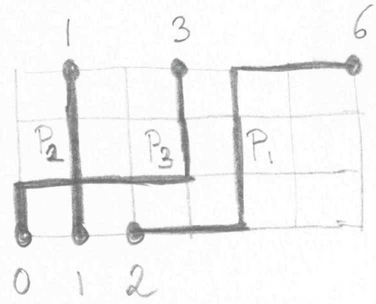


$= P$

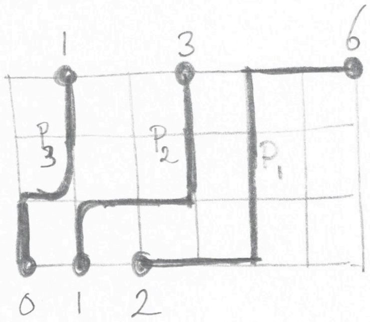
$$\text{weight}(P) = \prod_i x_i^{\#\text{horizontal steps at height } i} := x^P$$

Enlarge the set of paths by considering all paths from $(0,1,\dots,k)$ to $(\lambda_k, \lambda_{k-1}+1, \dots, \lambda_1+k-1)$

E.g.



$\sigma = (1,3,2)$



$\sigma = (1,2,3)$

By assigning the sign $\mathrm{sgn}(\sigma)$ to each set of paths according to the permutation σ encoding the arrival order, and by noting the intersecting sets of paths can be paired according to a flip in the first crossing (so that pairs have opposite sign) only nonintersecting sets of path contribute.

Hence

$$S_\lambda(x_1,\dots,x_n) = \sum_{P \text{ nonintersecting}} x^P$$

$$= \sum_{\text{all } P} \operatorname{sgn}(P)\, x^P$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sum_{\substack{\text{all } P \text{ from} \\ k-\sigma_i \mapsto \lambda_i + k - i \\ 1 \le i \le k}} x^P$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} h_{(\lambda_i + \cancel{k} - i) - \cancel{k} - \sigma_i}(x_1,\dots,x_n)$$

$$= \det_{1 \le i,j \le k} \left(h_{\lambda_i + j - i}(x_1,\dots,x_n) \right)$$

$\square$

<u>Theorem</u> (Cauchy identity)

$$\sum_\lambda s_\lambda[X]\, s_\lambda[Y] = \sigma_1[XY]$$

<u>Pf</u> There is a beautiful proof using RSK, but we have no time for that. Instead we will use the Jacobi-Trudi identity.

Let $X = \sum_{i=1}^n x_i$ and $Y = \sum_{i=1}^n y_i$.

Then $\sigma_1[XY] = \sum_\alpha h_\alpha[X]\, y^\alpha$; $h_\alpha := h_{\alpha_1} \cdots h_{\alpha_n}$

$\alpha = (\alpha_1, \ldots, \alpha_n)$ a (weak) composition

Vandermonde determinant

$$\sum_{w \in S_n} \mathrm{sgn}(w)\, w(y^\delta) = \Delta(Y)$$

$$\sigma_1[XY] = \frac{1}{\Delta(Y)} \sum_{w \in S_n} \sum_\alpha \mathrm{sgn}(w)\, h_\alpha[X]\, y^{\alpha + w(\delta)}$$

$\beta := \alpha + w(\delta) - \delta$

$$= \frac{1}{\Delta(Y)} \sum_\beta y^{\beta+\delta} \left(\sum_{w \in S_n} \mathrm{sgn}(w)\, h_{\beta+\delta-w(\delta)}[X] \right)$$

JT

$$= \frac{1}{\Delta(Y)} \sum_\beta y^{\beta+\delta}\, s_\beta[X]$$

Since $s_\beta = 0$ unless $\beta = w(\lambda+\delta) - \delta$, we may replace $\sum_\beta \mapsto \sum_\lambda \sum_{w \in S_n}$

$$= \frac{1}{\Delta(Y)} \sum_\lambda s_\lambda[X] \overbrace{\sum_{w \in S_n} \mathrm{sgn}(w)\, y^{w(\lambda+\delta)}}^{\Delta(Y) s_\lambda[Y]}$$

□

<u>Corollary</u> $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

<u>Lemma</u> $\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$

<u>Pf</u> Let the <u>Littlewood-Richardson</u> coeffs $c^\lambda_{\mu\nu}$ be defined as $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$

Then $$\sum_{\lambda,\mu} s_{\lambda/\mu}[X]\, s_\mu[Y]\, s_\lambda[Z]$$

$$= \sum_\lambda s_\lambda[X+Y]\, s_\lambda[Z]$$

$$= \sigma_1[(X+Y)Z]$$

$$= \sigma_1[XZ]\,\sigma_1[YZ]$$

$$= \sum_{\mu,\nu} s_\nu[X]\, s_\nu[Z]\, s_\mu[Y]\, s_\mu[Z]$$

$$= \sum_{\lambda,\mu,\nu} c^\lambda_{\mu\nu}\, s_\nu[X]\, s_\mu[Y]\, s_\lambda[Z]$$

Equating coefficients of $s_\mu[Y]s_\lambda[Z]$ yields

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$$

Finally

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \sum_\omega c^\lambda_{\mu\omega} \langle s_\omega, s_\nu \rangle = c^\lambda_{\mu\nu}$$

and

$$\langle s_\lambda, s_\mu s_\nu \rangle = \sum_\omega c^\omega_{\mu\nu} \langle s_\lambda, s_\omega \rangle = c^\lambda_{\mu\nu} \qquad \square$$

<u>Theorem</u> (Pieri rule)

$$h_r s_\mu = \sum_{\substack{\lambda \supset \mu \\ |\lambda/\mu| = r}} s_\lambda$$

<u>Pf</u> $\sum_\mu \sum_{r\geq 0} z^r h_r[X] s_\mu[X] s_\mu[Y]$

$$= \sigma_z[X]\sigma_1[XY] = \sigma_1[X(Y+z)]$$

$$= \sum_\lambda s_\lambda[X] s_\lambda[Y+z]$$

$$\underset{\text{branching rule}}{=} \sum_{\lambda} s_\lambda[X] \sum_{\mu < \lambda} z^{|\lambda/\mu|} s_\mu[Y]$$

Equating coefficients of $z^r s_\mu[Y]$ yields the Pieri rule. $\square$

<u>Remark</u> The branching and Pieri rule may be regarded as dual. This can be made even more precise using vertex operators.

For $n \in \mathbb{Z}$ let $\alpha_{-n} : \Lambda \to \Lambda$ be the linear operator which adds a borderstrip/ribbon of size n is all possible ways to a Schur function, where each such strip is weighted by $(-1)^{\text{height}(\text{b.s.})}$

Eg: $$\alpha_{-3}(s_{(3,1)}) = \underbrace{s_{(6,1)}}_{h=1-1=0} - \underbrace{s_{(3,2,2)}}_{h=2-1=1} + \underbrace{s_{(3,1,1,1,1)}}_{h=3-1=2}$$

$$\alpha_2(s_{(2,2)}) = s_{(2)} - s_{(1,1)}$$

Then the α_n satisfy the commutation relations of the <u>Heisenberg algebra</u>:

$$[\alpha_n, \alpha_m] = n\,\delta_{n,-m}$$

E.g. $\alpha_{-2}\alpha_{-1} S_{\ydiagram{1}}$

$$= \alpha_{-2}\left(S_{\ydiagram{2}} + S_{\ydiagram{1,1}}\right)$$

$$= \left(S_{\ydiagram{4}} + \cancel{S_{\ydiagram{2,2}}} - S_{\ydiagram{2,1,1}}\right) + \left(S_{\ydiagram{3,1}} - \cancel{S_{\ydiagram{2,2}}} - S_{\ydiagram{1,1,1,1}}\right)$$

$\alpha_{-1}\alpha_{-2} S_{\ydiagram{1}}$

$$= \alpha_{-1}\left(S_{\ydiagram{3}} - S_{\ydiagram{1,1,1}}\right)$$

$$= S_{\ydiagram{4}} + S_{\ydiagram{3,1}} - S_{\ydiagram{2,1,1}} - S_{\ydiagram{1,1,1,1}}$$

Using the α_n we can define the vertex operators

$$\Gamma_{\pm}(z) = \exp\left(\sum_{n\geq 1} \frac{z^n}{n}\,\alpha_{\pm n}\right)$$

E.g. $$\Gamma_-(z) = 1 + \alpha_{-1} z + \frac{z^2}{2}(\alpha_{-2} + \alpha_{-1}^2) + \frac{z^3}{6}(2\alpha_{-3} + 3\alpha_{-2}\alpha_{-1} + \alpha_{-1}^3) + \cdots$$

The vertex operators satisfy the commutation relation

$$\Gamma_+(w)\Gamma_-(z) = \frac{1}{1-zw}\,\Gamma_-(z)\Gamma_+(w)$$

Moreover,

$$\Gamma_-(z)\, s_\mu[X] = \sigma_z[X]\, s_\mu[X] \quad \text{(Pieri)}$$

$$\Gamma_+(z)\, s_\mu[X] = s_\mu[X+z] \quad \text{(branching)}$$

6 Schur processes in less than 5 minutes

(For more, see Okounkov (2001), Okounkov-Reshetikhin (2003), Borodin-Rains (2006), and many subsequent papers, including works by our magnificent host Leo P.)

Let G be a finite group and consider (for simplicity) the set I_G of irreducible representations over $\mathbb{C}$.

(This set is in 1-1 correspondence with the set of conjugacy classes of G.)

From character theory it immediately follows that

$$\sum_{\rho \in I_G} (\dim \rho)^2 = |G|$$

Hence we can define the Plancherel measure on I_G by

$$\mu(\rho) = \frac{(\dim \rho)^2}{|G|}$$

For example, for $G = S_n$, we can label the irreps by $\lambda \vdash n$ and write

$$\mu(\lambda) = \frac{(f^\lambda)^2}{n!}$$

where $f^\lambda = |SYT(\lambda)|$, $SYT(\lambda)$ the set of <u>standard Young tableaux</u> of shape λ

E.g. $SYT(2,1) = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \right\}$.

By the Frame-Robinson-Thrall hook-length formula,

$$f^\lambda = \frac{n!}{\prod_{h \in \mathcal{H}} h}$$

multiset of hook-lengths

E.g. $f^{(2,1)} = \dfrac{3!}{1 \cdot 1 \cdot 3} = 2.$

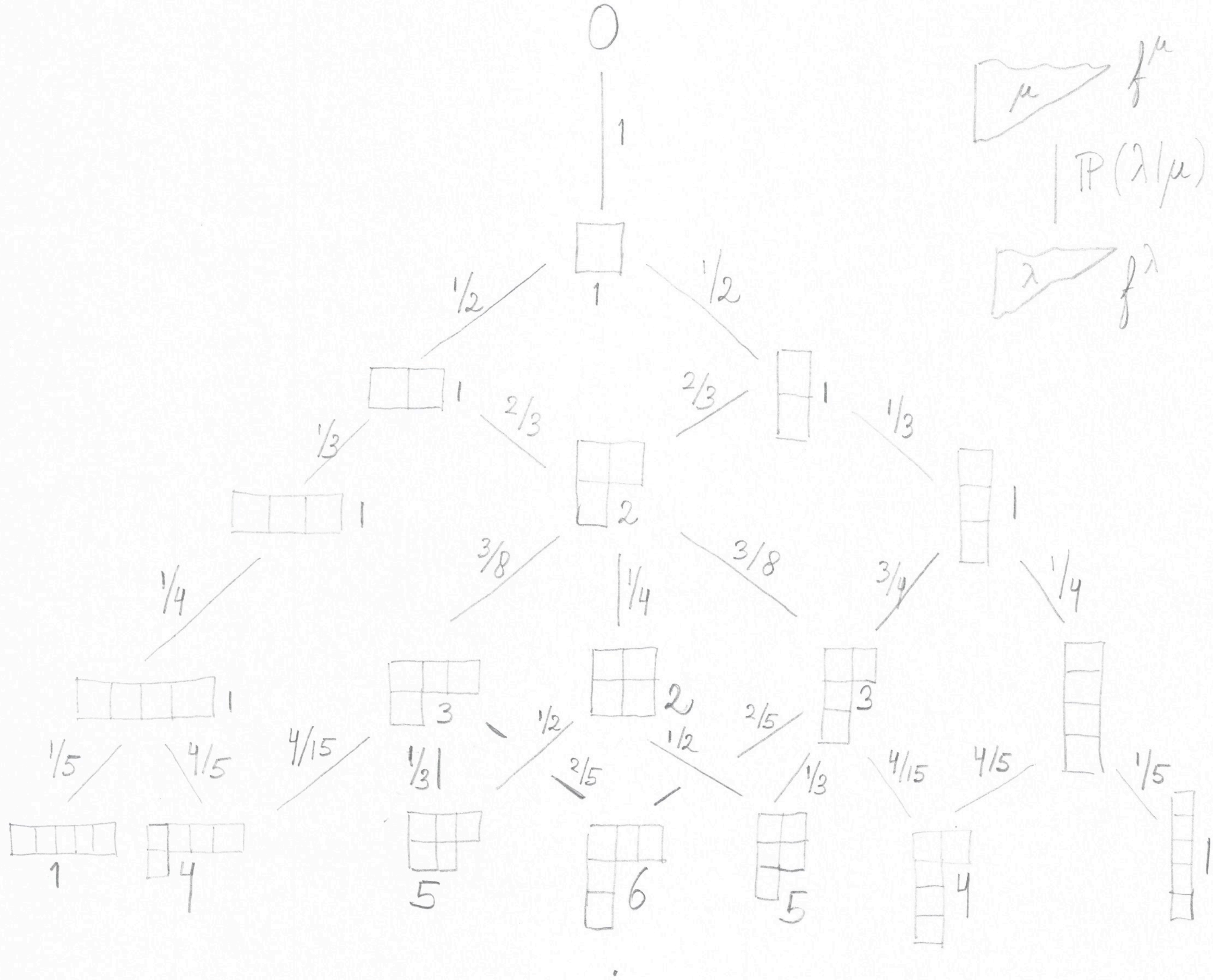
We may obviously view $\mu(\lambda)$ as a measure on the set of partitions of size n.

(This can be turned into a measure on all partition through 'Poissonisation' $\mu_{PPM}(\lambda) := e^{-\xi} \xi^{|\lambda|} \dfrac{(f^\lambda)^2}{|\lambda|!}$, $\xi > 0$)

Correspondingly, one can define the <u>Plancherel</u> (growth) <u>process</u> as a directed random walk on the <u>Young lattice</u>, with transition probabilities

$(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \dots)$, $|\lambda^{(n)}| = n$

$$\mathbb{P}\left(\lambda^{(n)} = \lambda \mid \lambda^{(n-1)} = \mu\right) = \frac{f^{\lambda}}{n\, f^{\mu}}$$



Eg. $\mathbb{E}(\ell(\lambda)) = \frac{67}{24} \approx 2.79 < \frac{20}{7}$ for uniform distribution. ($\lambda \vdash 5$)

$\mathbb{E}(f^{\lambda}) = \frac{149}{30} \approx 4.97 < \frac{26}{7}$ 〃

In the Plancherel process, one $\square$ is added to a partition at every time step. More generally, let $\omega = (\omega_1, \omega_2, \ldots)$ be a finite or infinite sequence with $\omega_i \in \{\prec, \succ, \prec', \succ'\}$ where $\mu \prec' \lambda$ iff $\mu' \prec \lambda'$ (i.e $\mu \prec' \lambda$ iff λ/μ is a vertical strip) and

$$\Lambda = (\underset{0}{\overset{\shortparallel}{\lambda^{(0)}}}, \lambda^{(1)}, \lambda^{(2)}, \ldots) \text{ a sequence of partitions}$$

such that $\lambda^{(i-1)} \omega_i \lambda^{(i)}$

Then the <u>Schur process</u> is the measure on ω-interlaced partitions, such that

$$\mathrm{Prob}(\Lambda) \propto \prod_{i \geq 1} x_i^{|\lambda^{(i)}| - |\lambda^{(i-1)}|}$$

Many variants are possible, e.g.,

$$\Lambda = (\mu = \lambda^{(-r)}, \lambda^{(1-r)}, \ldots, \lambda^{(-1)}, \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(s)} = \lambda)$$

etc

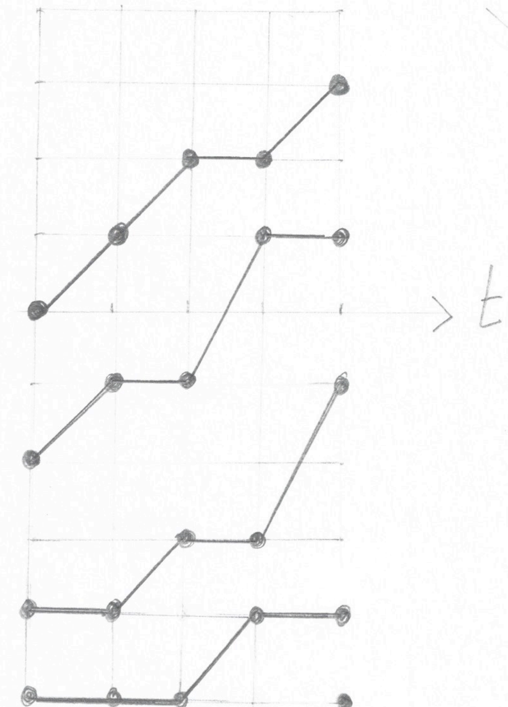
Examples

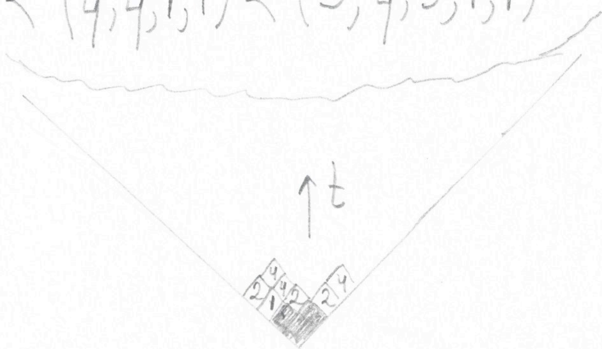
(1) $\omega = \{ \prec \}^n$

$$\Lambda = \{ \mu = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \dots \prec \lambda^{(n)} = \lambda \}$$

$$\mathrm{Prob}(\Lambda) = \frac{\prod_{i \geq 1} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|}}{s_{\lambda/\mu}(x_1, \dots, x_n)}$$

Eg $(2,1) \prec (3,2) \prec (4,2,1) \prec (4,4,1,1) \prec (5,4,3,1,1)$





The point configuration $\mathcal{L}(\Lambda) = \{ (t, \lambda_i^{(t)} - i) \}_{\substack{i \geq 1 \\ 0 \leq t \leq T}} \subseteq \{0, \dots, T\} \times \mathbb{Z}$

(2) $\omega = \{ \underbrace{<, <, \dots, <}_{r} \ \underbrace{>, \dots >, >}_{s} \}$

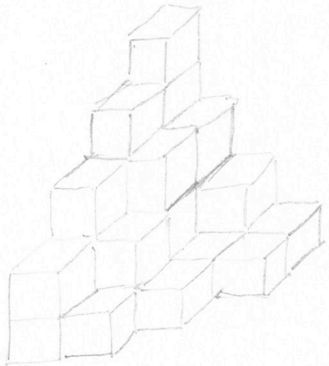
$$\Lambda = \{ 0 = \lambda^{(-r)} < \dots < \lambda^{(-1)} < \lambda^{(0)} > \lambda^{(1)} > \dots > \lambda^{(s)} = 0 \}$$

Eg (After Okounkov & Reshetikhin)

$$(2) < (3,1) < (4,2) < (5,3,1) > (3,1) > (2,1) > (1)$$

$$\xleftarrow{-t} \quad \prod_i \Gamma_-(y_i)$$

$$\prod_i \Gamma_-(x_i) \uparrow \qquad t \uparrow \quad \begin{array}{|c|c|c|c|l} \hline 5 & 3 & 2 & 1 & \nwarrow y_1 \\ \hline 4 & 3 & 1 & 1 & \nwarrow y_2 \\ \hline 3 & 2 & 1 & \multicolumn{2}{l}{\nwarrow y_s} \\ \cline{1-3} 2 & 1 & \multicolumn{3}{l}{\nwarrow x_r} \\ \cline{1-2} \multicolumn{1}{c}{\nwarrow x_1} & \multicolumn{1}{c}{\nwarrow x_2} \end{array}$$



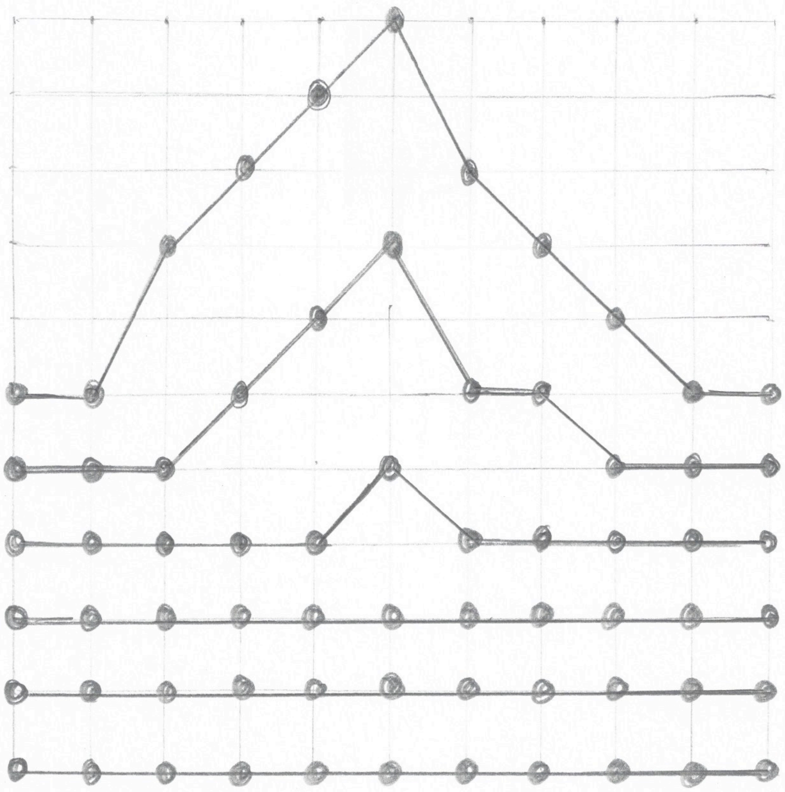
plane partition confined in a "box" $B(r,s,\infty)$.

MacMahon $$\sum_{\pi \in B(r,s,\infty)} q^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1}{1-q^{i+j-1}}$$

$$\sum_{\Lambda} \cdots = \sum_{\lambda^{(0)}} S_{\lambda^{(0)}}(x_1, \dots, x_r)\, S_{\lambda^{(0)}}(y_1, \dots, y_s)$$

$$\underset{\substack{\uparrow \\ \text{Cauchy}}}{=} \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1}{1-x_i y_j} \xrightarrow[y_i \mapsto q^{s-i+1/2}]{x_i \mapsto q^{r-i+1/2}} \text{MacMahon}$$

This time we have the point configuration



$t=0 \quad \longrightarrow t$

(3) $\omega = \underbrace{<,<',<,<',\ldots,<,<'}_{r\text{ times}} \; \underbrace{>'>\ldots>'>>'>}_{s\text{ times}}$

Eg $\quad 0 < (1) <' (2) < (2,2) \underset{x_{-2}}{<'} (3,3) \underset{x_{-1}}{<} (3,3,2) \underset{x_1}{>'} (2,2,1) \underset{x_2}{>}$

$> 21 >' (1,1) > (1) >' 0$

Pyramid partitions $\quad x_i = x_{-i} = q^{i-1/2}$

$$\sum_{\pi} q^{|\pi|} = \prod_{i\geq 1} \frac{(1+q^{2i-1})^{2i-1}}{(1-q^{2n})^{2n}} \qquad \text{(Young)}$$

Theorem (Okounkov-Reshetikhin '03)

The Schur process is a determinantal point process.

$$\mathbb{P}\left(\lambda^{(i_k)} \text{ has a } \bullet \text{ at } y_k \text{ for } 1 \le k \le n\right)$$

$$= \det_{1 \le l, m \le n} K(i_l, k_l; i_m, k_m)$$

$$K(i,k;j,l) = \begin{cases} \left[\dfrac{z^k}{w^l}\right] \dfrac{\phi(z; x_1,\dots,x_n; \omega_1,\dots,\omega_n; i)}{\phi(w; x_1,\dots,x_n; \omega_1,\dots,\omega_n; j)} \dfrac{(zw)^{1/2}}{z-w} & i \le j \\ -\left[\dfrac{z^k}{w^l}\right] \dfrac{\phi(\quad '' \quad ; j)}{\phi(\quad '' \quad ; i)} \dfrac{(zw)^{1/2}}{w-z} & i > j \end{cases}$$

$$\phi(z; x_1,\dots,x_n; \omega_1,\dots,\omega_n; i)$$

$$= \prod_{\substack{j \le i \\ \omega_j = <}} \sigma_z[x_j] \prod_{\substack{j \le i \\ \omega_j = <'}} \sigma_z^{-1}[-x_j] \prod_{\substack{j > i \\ \omega_j = >}} \sigma_{z^{-1}}^{-1}[x_j] \prod_{\substack{j > i \\ \omega_j = >'}} \sigma_{z^{-1}}[-x_j]$$