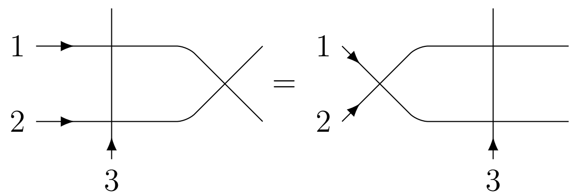
QUANTUM INTEGRABILITY AND SYMMETRIC POLYNOMIALS

1. Exercise session 1

1.1. **Low-temperature expansion of the antiferroelectric six-vertex model.** Consider the six-vertex model with the usual Boltzmann weights a, b, c on a torus of size $K \times L$. We assume that K and L are even, and consider the regime where $c \gg a, b$.

- Describe the two ground states of the model, i.e., the configurations of the model with maximal Boltzmann weight.
- Use graphical notations to verify that to ninth order in a/c, b/c, the partition function is given by
$$\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V+1)a^4b^4 + Va^2b^2(a^4+b^4) + \cdots,$$
where we have set $c = 1$ for convenience, and $V := KL$.
- Explain the symmetry of Z in a, b from the symmetries of the model.

1.2. **Determination of the six-vertex $\boldsymbol{R}$-matrix from $\boldsymbol{RLL}$ relations.** We encode the Boltzmann weights $a, b, c > 0$ of the six-vertex model into an L-matrix, and similarly for another set of weights a', b', c':
$$L = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}, \qquad L' = \begin{pmatrix} a' & & & \\ & b' & c' & \\ & c' & b' & \\ & & & a' \end{pmatrix}.$$
We want to find conditions on the weights a, b, c, a', b', c' such that there exists an invertible R-matrix of the same form
$$R = \begin{pmatrix} a'' & & & \\ & b'' & c'' & \\ & c'' & b'' & \\ & & & a'' \end{pmatrix}$$
such that the RLL relations hold, namely
$$R_{12}L_{13}L'_{23} = L'_{23}L_{13}R_{12},$$



- Show that due to line conservation out of these $2^6 = 64$ equations only $\sum_{k=0}^{3}\binom{3}{k}^2 = 20$ are nontrivial, which due to parity symmetry come in pairs of identical equations.
- Noting that the left and right hand sides are exchanged by 180° rotation of the corresponding pictures, conclude that out of the ten equations, four correspond to

180° rotation invariant boundary conditions and are therefore automatically satisfied, while six come in pairs of identical equations and read:

$$\begin{aligned} ab'c'' + cc'b'' &= ba'c'', \\ ac'b'' + cb'c'' &= bc'a'', \\ ac'c'' + cb'b'' &= ca'a''. \end{aligned}$$

Draw the corresponding configurations.

- Eliminating a'', b'', c'', show that a solution for R exists only if $\Delta(a,b,c) = \Delta(a',b',c')$ with
$$\Delta(a,b,c) = \frac{a^2+b^2-c^2}{2ab}.$$
- Show that if $\Delta(a,b,c) = \Delta(a',b',c')$ then one also has $\Delta(a'',b'',c'') = \Delta(a,b,c)$, i.e., the R-matrix is of the same form as the L-matrices.
- Assume $\Delta \neq \pm 1$. Recall that if one writes $\Delta = \frac{q+q^{-1}}{2}$, then the Boltzmann weights can be parameterized as

$$\begin{aligned} a(z) &= q\,z - q^{-1}z^{-1}, \\ b(z) &= z - z^{-1}, \\ c(z) &= q - q^{-1}, \end{aligned} \tag{1}$$

up to overall normalization. Here z is the *spectral parameter*.

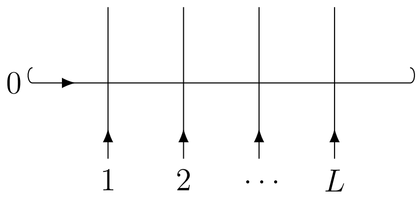
Show that if L is parameterized as above and L' similarly with z replaced by z', then R is of the same form with spectral parameter $z'' = z/z'$.

1.3. **XXZ Hamiltonians.** Parameterize the six-vertex R-matrix for $\Delta = \frac{q+q^{-1}}{2} \neq \pm 1$ as in the previous exercise:

$$R(z) = \begin{pmatrix} a(z) & & & \\ & b(z) & c(z) & \\ & c(z) & b(z) & \\ & & & a(z) \end{pmatrix}$$

with weights given by (1). The (homogeneous) transfer matrix is defined by

$$T(z) = \mathrm{tr}_0(R_{0L}(z)\dots R_{01}(z)) =$$



- Show that up to normalization $T(1)$ is the *translation operator* U, i.e., $U\, e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_L} = e_{k_L} \otimes e_{k_1} \otimes \cdots \otimes e_{k_{L-1}}$ where e_+, e_- are standard basis vectors.
- Show that up to an additive constant and normalization $(\log T)'(1) = \frac{dT}{dz}T^{-1}|_{z=1}$ is the *XXZ Hamiltonian*

$$H_{\mathrm{XXZ}} = \sum_{i=1}^{L} \left(2\,(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) + \Delta\,\sigma_i^z\sigma_{i+1}^z \right),$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where periodic boundary conditions are implicit ($L+1 \equiv 1$), and $\sigma_i^\bullet$ means $\sigma^\bullet$ acting on the i^{th} factor of the tensor product $(\mathbb{C}^2)^{\otimes L}$.

Hint. One may show first that $\frac{dT}{dz}T^{-1}|_{z=1} = (q-q^{-1})^{-1}\sum_{i=1}^L \frac{d\check{R}_{i\,i+1}}{dz}|_{z=1}$ where $\check{R}_{i\,i+1}(z) = P_{i\,i+1}R_{i\,i+1}(z)$ and $P_{i\,i+1}$ exchanges the spins at sites i and $i+1$, i.e., $P_{i\,i+1}\, e_{k_1}\otimes\cdots\otimes e_{k_i}\otimes e_{k_{i+1}}\otimes\cdots\otimes e_{k_L} = e_{k_1}\otimes\cdots\otimes e_{k_{i+1}}\otimes e_{k_i}\otimes\cdots\otimes e_{k_L}$.

- (*Optional*) Compute the second XXZ Hamiltonian $(\log T)''(1)$.

1.4. **The 1D Ising model.** With the same notations as in the previous exercise, consider the Hamiltonian

$$H_{\text{Ising}} = J\sum_{i=1}^L \sigma_i^z\sigma_{i+1}^z + h\sum_{i=1}^L \sigma_i^z.$$

- How is H_{Ising} related to H_{XXZ} of the previous exercise?
- Show that $Z = \operatorname{tr}(e^{-\beta H_{\text{Ising}}})$ is the partition function of the *classical one-dimensional Ising model*:

$$Z = \sum_{\sigma\colon \mathbb{Z}/L\mathbb{Z}\to\{\pm1\}} e^{-\beta J\sum_{i=1}^L \sigma_i\sigma_{i+1} - \beta h\sum_{i=1}^L \sigma_i}.$$

- Denoting $K = -\beta J$, $B = -\beta h$, show that

$$Z = \operatorname{tr}(T^L), \qquad T = \begin{pmatrix} e^{K+B} & e^{-K} \\ e^{-K} & e^{K-B}\end{pmatrix},$$

  where T plays the role of a *one-dimensional* transfer matrix.
- Conclude that

$$Z = \Lambda_+^L + \Lambda_-^L, \qquad \Lambda_\pm = e^K\cosh B \pm \sqrt{e^{2K}\sinh^2 B + e^{-2K}}.$$

- Compute $\lim_{L\to\infty}\frac{\log Z}{L}$. Is there any phase transition in temperature, i.e., in the parameter β (the inverse temperature) as it varies from 0 to $+\infty$?

2. Exercise session 2

2.1. **The q-determinant.** Consider the *Yang–Baxter (bi)algebra* associated to the six-vertex model R-matrix, with the standard notation $\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ for the generating series of its generators.

- Prove the series of equalities

$$\begin{aligned} \mathrm{qdet}(z) &:= A(qz)D(z) - B(qz)C(z) \\ &= D(qz)A(z) - C(qz)B(z) \\ &= A(z)D(qz) - C(z)B(qz) \\ &= D(z)A(qz) - B(z)C(qz). \end{aligned}$$

- Prove that $\mathrm{qdet}(z)$ is central, i.e, commutes with all elements of the Yang–Baxter (bi)algebra, and prove that it is group like, i.e., $\Delta(\mathrm{qdet}(z)) = \mathrm{qdet}(z) \otimes \mathrm{qdet}(z)$.

2.2. **Commutation of twisted transfer matrices.** Consider as above the Yang–Baxter (bi)algebra associated to the six-vertex model R-matrix. Recall that its defining relations, the RTT relations, can be written in components as sixteen relations for its generators $A(z)$, $B(z)$, $C(z)$, $D(z)$.

- Write explicitly the components of the RTT relations involving $A(z)$ and $D(z)$ that you will need for the following part.
- Defining

$$T_\kappa(z) = A(z) + \kappa D(z), \qquad \kappa \in \mathbb{C},$$

  conclude that $[T_\kappa(z), T_\kappa(z')] = 0$ for all z, z'.

  In other words, for a fixed κ, $T_\kappa(z)$ is the generating series for a commutative subalgebra of the Yang–Baxter algebra.

2.3. **Bethe Ansatz equations as pole cancellations.** Consider the six-vertex model with periodic boundary conditions, and its transfer matrix $T(z) = A(z) + D(z)$ acting on $(\mathbb{C}^2)^{\otimes L}$.

- Consider an eigenvector $|\Psi\rangle$ of $T(z)$, with eigenvalue

$$T(z)\,|\Psi\rangle = t(z)\,|\Psi\rangle\,.$$

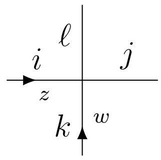
  As a function of z, what can be said about $t(z)$?
- Now assume $|\Psi\rangle$ is a Bethe vector. Write the formula expressing the eigenvalue $t(z)$ as a function of the Bethe roots $z_1, \ldots, z_M$. What is its dependence on z? Comparing with the previous part, conclude that the residues of $t(z)$ at the would-be poles of this formula must vanish. Compute these residues and compare with Bethe Ansatz equations.

2.4. **Energy/momentum of XXZ eigenvectors.**

- Using the trace identities, cf. exercise 1.3, compute the momentum and XXZ energy of a Bethe vector in terms of the Bethe roots. (Recall that the shift operator U is unitary, so its eigenvalues are of the form e^{ip} where $p \in \mathbb{R}/2\pi\mathbb{Z}$ is the momentum.)
- Argue that these states can be viewed as consisting of quasiparticles called magnons, where each magnon can be associated with one Bethe root.
- How does the isotropy at $\Delta = 1$ show up in the spectrum?

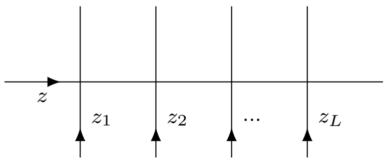
2.5. **Yang–Baxter algebra representations and inhomogeneous monodromy matrix.** We recall that a *representation* of an algebra $\mathcal{A}$ is the data of a vector space V and an algebra morphism $\rho : \mathcal{A} \to \mathrm{End}(V)$, i.e., a linear map preserving the multiplication. If $\mathcal{A}$ is a *bialgebra*, then one can take tensor products of representations using the coproduct Δ: given (V_1, ρ_1) and (V_2, ρ_2), define the representation $(V_1 \otimes V_2, \rho_{1\otimes 2})$ by $\rho_{1\otimes 2}(a) = (\rho_1 \otimes \rho_2)\Delta(a)$ for all $a \in \mathcal{A}$.

Consider the Yang–Baxter bialgebra $\mathcal{A}$ with generators $(\hat{\mathcal{T}}_i^j(z))_{i,j=1,\dots,n}$ associated to an R-matrix $R(z) = (R_{ik}^{j\ell}(z))_{i,j,k,\ell=1,\dots,n}$:

$R_{ik}^{j\ell}(z/w) =$ 

satisfying the Yang–Baxter equation. One may limit oneself to the case of the six-vertex model R-matrix, with $n = 2$: $\hat{\mathcal{T}}_1^1(z) = \hat{A}(z)$, $\hat{\mathcal{T}}_1^2(z) = \hat{B}(z)$, $\hat{\mathcal{T}}_2^1(z) = \hat{C}(z)$, $\hat{\mathcal{T}}_2^2(z) = \hat{D}(z)$.

- Show that $\hat{\mathcal{T}}_i^j(z) \overset{\rho_z}{\longmapsto} (R_{ik}^{j\ell}(z/w))_{k,\ell=1,\dots,n}$ defines a representation of $\mathcal{A}$ on the vector space $\mathbb{C}^n$. This representation is often denoted $\mathbb{C}^n(w)$.
- Define the *inhomogeneous* monodromy matrix

  $\mathcal{T}(z; z_1, \dots, z_L) = R_{0L}(z/z_L) \dots R_{01}(z/z_1) =$ 

  Note that the usual (homogeneous) monodromy matrix is the special case $\mathcal{T}(z) = \mathcal{T}(z; 1, \dots, 1)$.

  Show that $\hat{\mathcal{T}}_i^j(z) \mapsto \mathcal{T}_i^j(z; z_1, \dots, z_L)$ is the tensor product representation $\mathbb{C}^n(z_1) \otimes \mathbb{C}^n(z_2) \otimes \cdots \otimes \mathbb{C}^n(z_L)$ of $\mathcal{A}$.

3. Exercise session 3

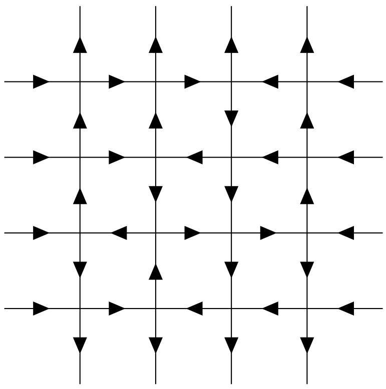
3.1. **Alternating Sign Matrices, tableaux and Gelfand–Tseytlin patterns.** Recall that there is a bijection between six-vertex configurations with Domain Wall Boundary Conditions (DWBC) and Alternating Sign Matrices (ASMs).

- As a warm up find the DWBC configuration, both in the arrow picture and the path picture, and the domino tilings of the Aztec diamond corresponding to the ASM

$$\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}$$

The object of this exercise is to uncover more bijections. In the remainder one may use the following working example:

$$\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}$$



and provide the image of the example under the various bijections below.

A *Gelfand–Tseytlin pattern* is a triangle of (integer) numbers $(\lambda_j^{(i)})_{1\leq j\leq i, 1\leq i\leq n}$ such that the inequalities hold $\lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_{j+1}^{(i+1)}$; i.e.,

$$\begin{array}{ccccccc} \lambda_1^{(4)} & & \lambda_2^{(4)} & & \lambda_3^{(4)} & & \lambda_4^{(4)} \\ \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & \\ & \lambda_1^{(3)} & & \lambda_2^{(3)} & & \lambda_3^{(3)} & \\ & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & & \\ & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & \\ & & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & & & \\ & & & \lambda_1^{(1)} & & & \end{array}$$

A *strict* Gelfand–Tseytlin pattern (a.k.a. monotone triangle) is defined identically, except we further impose the strict inequality $\lambda_j^{(i+1)} > \lambda_{j+1}^{(i+1)}$:

$$
\begin{array}{ccccccccccccc}
\lambda_1^{(4)} & & > & & \lambda_2^{(4)} & & > & & \lambda_3^{(4)} & & > & & \lambda_4^{(4)} \\
& \geq & & \leq & & \geq & & \leq & & \geq & & \leq & \\
& & \lambda_1^{(3)} & & > & & \lambda_2^{(3)} & & > & & \lambda_3^{(3)} & & \\
& & & \geq & & \leq & & \geq & & \leq & & & \\
& & & & \lambda_1^{(2)} & & > & & \lambda_2^{(2)} & & & & \\
& & & & & \geq & & \leq & & & & & \\
& & & & & & \lambda_1^{(1)} & & & & & &
\end{array}
$$

Finally, a Semi-Standard Young tableau (SSYT) is a filling $(T_{i,j})_{(i,j)\in Y}$ of a Young diagram Y with positive integers such $T_{i,j} < T_{i+1,j}$ and $T_{i,j} \leq T_{i,j+1}$, i.e.,

1	2	2	3
3	3		
4			

We identify Young diagram with partitions – the example above is $(4, 2, 1)$.

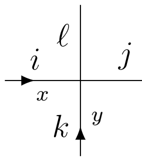
- Given a DWBC configuration of size n, define a triangular array of numbers as follows. The numbers are the columns of up-pointing arrows; more precisely, $\lambda_j^{(i)}$ is the column number (counted left to right from 1 to n) of the j^{th} up-arrow (counted from the *right*) of the i^{th} row (counted from the bottom).
  Show that the resulting triangular array is a strict Gelfand–Tseytlin pattern, and that this provides a bijection between DWBC configurations of size n and strict Gelfand–Tseytlin patterns with top row $(n, \ldots, 2, 1)$.
- Given a (not necessarily strict) Gelfand–Tseytlin pattern $(\lambda_j^{(i)})$ with nonnegative entries, one can produce a tableau as follows: each row $\lambda^{(i)}$ of the pattern is a partition, which can be drawn as a Young diagram; we obtain this way a sequence of Young diagrams which is weakly decreasing w.r.t. inclusion. In turn, this gives a tableau of the partition of the top row $\lambda^{(n)}$ as follows: a box of the Young diagram has label $i \in \{1, \ldots, n\}$ iff it belongs to (the Young diagram of) $\lambda^{(i)}$ but not to $\lambda^{(i-1)}$ (with the convention that $\lambda^{(0)}$ is the empty partition).
  Show that the resulting tableau is semi-standard, and that this forms a bijection between Gelfand–Tseytlin patterns with fixed first row $\lambda^{(n)}$ and the SSYTs of the partition $\lambda^{(n)}$ with labels in $\{1, \ldots, n\}$.
- Given a DWBC configuration, apply successively the bijections of the two previous questions to produce a SSYT $(T_{i,j})$. What is its shape? Define a new triangular array by $(\lambda_j^{(i)})$ by $\lambda_j^{(i)} = T_{i+1-j,n+1-i}$. This corresponds to rotating and deforming the SSYT:

$$
\begin{array}{|c|c|c|c|}
\hline
T_{1,1} & T_{1,2} & T_{1,3} & T_{1,4} \\
\hline
T_{2,1} & T_{2,2} & T_{2,3} & \multicolumn{1}{c}{} \\
\cline{1-3}
T_{3,1} & T_{3,2} & \multicolumn{2}{c}{} \\
\cline{1-2}
T_{4,1} & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}
\quad \mapsto \quad
\begin{array}{ccccccc}
T_{4,1} & & T_{3,1} & & T_{2,1} & & T_{1,1} \\
& T_{3,2} & & T_{2,2} & & T_{1,2} & \\
& & T_{2,3} & & T_{1,3} & & \\
& & & T_{1,4} & & &
\end{array}
$$

Show that $(\lambda_j^{(i)})$ is again a strict Gelfand–Tseytlin pattern, and that it is associated via the first bijection to the $\pi/2$ clockwise rotation of the original DWBC configuration with all arrows reversed (or equivalently, to the $\pi/2$ clockwise rotation of the original ASM).

3.2. **NilHecke solution of Yang–Baxter equation and Schubert polynomials.** Let $r \in \mathbb{Z}_{>0}$, and consider the following *rational* R-matrix:

$$R_{ik}^{j\ell}(x-y) =$$

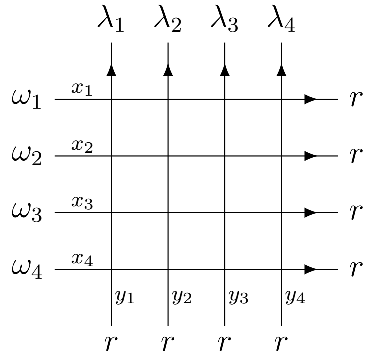


$$= \begin{cases} 1 & i = \ell, \quad k = j \\ x - y & i = j < k = \ell \\ 0 & \text{else} \end{cases}, \qquad 1 \le i, j, k, \ell \le r$$

(where as usual, superscripts are row indices and subscripts are column indices). Viewed as an operator, it acts on $\mathbb{C}^r \otimes \mathbb{C}^r$.

Given a n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \{1, \dots, r\}^n$, we define ω to be its "sort", i.e., the only weakly increasing permutation of λ. E.g., if $\lambda = (2, 1, 3, 2)$, then $\omega = (1, 2, 2, 3)$. We define the *Schubert polynomial* S_λ associated to λ to be the following partition function: (on this example, $n = 4$)

$$S_\lambda =$$



- Show that R satisfies the Yang–Baxter equation with additive spectral parameters.
- When $r = 2$, what is the R-matrix? Still at $r = 2$, what is S_λ when all ys are 0?
- For general r, show that S_λ is a *homogeneous polynomial* in the xs and ys. What is its degree?

  Denote $I_a = \{i : \omega_i = a\}$ for $a = 1, \dots, r$.
- Show that S_λ does not depend on the x_i, $i \in I_r$.
- Show that for each $a \in \{1, \dots, r\}$, S_λ is invariant by permutation of the x_i, $i \in I_a$.
- Given a n-tuple λ, define its *standardization* μ to be the unique n-tuple such that each integer in $\{1, \dots, n\}$ occurs once (i.e., μ is a permutation) and for all $i < j$ $\lambda_i \le \lambda_j$ iff $\mu_i < \mu_j$. E.g., if $\lambda = (2, 1, 3, 2)$ then $\mu = (2, 1, 4, 3)$.

  Show that $S_\lambda = S_\mu$ (where S_μ is defined by choosing the value of r to be n).
- Define the *inversion code* $\underline{\lambda}$ of a n-tuple λ to be the sequence

$$\underline{\lambda} = (\#\{j > i : \ \lambda_j < \lambda_i\})_{i=1,\dots,n}$$

  e.g., if $\lambda = (5, 2, 1, 3, 2)$ then $\underline{\lambda} = (4, 1, 0, 1, 0)$. Show that if $\underline{\lambda}$ is weakly decreasing, then

$$S_\lambda = \prod_{j=1}^{n} \prod_{i=1}^{\underline{\lambda}_j} (x_i - y_j)$$

  If $r = 2$, what are the λ satisfying this condition?