1. Exercise session 1

1.1. Low-temperature expansion of the antiferroelectric six-vertex model. Consider the six-vertex model with the usual Boltzmann weights $a, b, c$ on a torus of size $K \times L$. We assume that $K$ and $L$ are even, and consider the regime where $c \gg a, b$.

- Describe the two ground states of the model, i.e., the configurations of the model with maximal Boltzmann weight.
- Use graphical notations to verify that to ninth order in $a/c$, $b/c$, the partition function is given by
  \[ \frac{1}{2} Z = 1 + V a^2 b^2 + V a^2 b^2 (a^2 + b^2) + \frac{1}{2} V (V+1) a^4 b^4 + V a^2 b^2 (a^4 + b^4) + \cdots, \]
  where we have set $c = 1$ for convenience, and $V := KL$.
- Explain the symmetry of $Z$ in $a, b$ from the symmetries of the model.

1.2. Determination of the six-vertex $R$-matrix from RLL relations. We encode the Boltzmann weights $a, b, c > 0$ of the six-vertex model into an $L$-matrix, and similarly for another set of weights $a', b', c'$:

\[
L = \begin{pmatrix} a & b & c \\ c & b & a \\ b & c & a \end{pmatrix}, \quad L' = \begin{pmatrix} a' & b' & c' \\ c' & b' & a' \\ b' & c' & a' \end{pmatrix}.
\]

We want to find conditions on the weights $a, b, c, a', b', c'$ such that there exists an invertible $R$-matrix of the same form

\[
R = \begin{pmatrix} a'' & b'' & c'' \\ c'' & b'' & a'' \end{pmatrix}
\]

such that the RLL relations hold, namely

\[
R_{12} L_{13} L'_{23} = L'_{23} L_{13} R_{12},
\]

- Show that due to line conservation out of these $2^6 = 64$ equations only $\sum_{k=0}^{2} (\binom{3}{k})^2 = 20$ are nontrivial, which due to parity symmetry come in pairs of identical equations.
- Noting that the left and right hand sides are exchanged by $180^\circ$ rotation of the corresponding pictures, conclude that out of the ten equations, four correspond to
180° rotation invariant boundary conditions and are therefore automatically satisfied, while six come in pairs of identical equations and read:

\[
\begin{align*}
ab'c'' + cc'b'' &= ba'c'', \\
ac'b'' + cb'c'' &= bc'a'', \\
ac'c'' + cb'b'' &= ca'a''.
\end{align*}
\]

Draw the corresponding configurations.

- Eliminating \(a'', b'', c''\), show that a solution for \(R\) exists only if \(\Delta(a, b, c) = \Delta(a', b', c')\) with

\[
\Delta(a, b, c) = \frac{a^2 + b^2 - c^2}{2ab}.
\]

- Show that if \(\Delta(a, b, c) = \Delta(a', b', c')\) then one also has \(\Delta(a'', b'', c'') = \Delta(a, b, c)\), i.e., the \(R\)-matrix is of the same form as the \(L\)-matrices.

- Assume \(\Delta \neq \pm 1\). Recall that if one writes \(\Delta = \frac{q + q^{-1}}{2}\), then the Boltzmann weights can be parameterized as

\[
\begin{align*}
a(z) &= qz - q^{-1}z^{-1}, \\
b(z) &= z - z^{-1}, \\
c(z) &= q - q^{-1},
\end{align*}
\]

up to overall normalization. Here \(z\) is the spectral parameter.

Show that if \(L\) is parameterized as above and \(L'\) similarly with \(z\) replaced by \(z'\), then \(R\) is of the same form with spectral parameter \(z'' = z/z'\).

1.3. XXZ Hamiltonians. Parameterize the six-vertex \(R\)-matrix for \(\Delta = \frac{q + q^{-1}}{2} \neq \pm 1\) as in the previous exercise:

\[
R(z) = \begin{pmatrix} a(z) & b(z) & c(z) \\ c(z) & b(z) & a(z) \end{pmatrix}
\]

with weights given by (1). The (homogeneous) transfer matrix is defined by

\[
T(z) = tr_0(R_{0L}(z) \ldots R_{01}(z)) = 0 \quad 1 \quad 2 \quad \ldots \quad L
\]

- Show that up to normalization \(T(1)\) is the translation operator \(U\), i.e., \(U e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_L} = e_{k_L} \otimes e_{k_1} \otimes \cdots \otimes e_{k_{L-1}}\) where \(e_+, e_-\) are standard basis vectors.

- Show that up to an additive constant and normalization \((\log T)'(1) = \frac{dT}{dz}T^{-1}|_{z=1}\) is the XXZ Hamiltonian

\[
H_{XXZ} = \sum_{i=1}^{L} \left(2(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) + \Delta \sigma_i^z \sigma_{i+1}^z\right),
\]

where

\[
\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
where periodic boundary conditions are implicit ($L + 1 \equiv 1$), and $\sigma_i^\bullet$ means $\sigma^\bullet$ acting on the $i^{th}$ factor of the tensor product $(\mathbb{C}^2)^\otimes L$.

**Hint.** One may show first that $\frac{dT}{dz}T^{-1}|_{z=1} = (q - q^{-1})^{-1} \sum_{i=1}^L \frac{dR_{i+1}}{dz}|_{z=1}$ where $\bar{R}_{i+1}(z) = P_{i+1}R_{i+1}(z)$ and $P_{i+1}$ exchanges the spins at sites $i$ and $i + 1$, i.e., $P_{i+1} e_k \otimes \cdots \otimes e_k \otimes e_{k+1} \otimes \cdots e_{kL} = e_{k+1} \otimes \cdots \otimes e_{k+1} \otimes e_k \otimes \cdots \otimes e_{kL}$.

- (Optional) Compute the second XXZ Hamiltonian ($\log T''(1)$).

1.4. **The 1D Ising model.** With the same notations as in the previous exercise, consider the Hamiltonian

$$H_{\text{Ising}} = J \sum_{i=1}^L \sigma_i^z \sigma_{i+1}^z + h \sum_{i=1}^L \sigma_i^z.$$  

- How is $H_{\text{Ising}}$ related to $H_{\text{XXZ}}$ of the previous exercise?
- Show that $Z = \text{tr}(e^{-\beta H_{\text{Ising}}})$ is the partition function of the classical one-dimensional Ising model:

$$Z = \sum_{\sigma : \mathbb{Z}/L\mathbb{Z} \rightarrow \{\pm 1\}} e^{-\beta J \sum_{i=1}^L \sigma_i \sigma_{i+1}} - \beta h \sum_{i=1}^L \sigma_i.$$  

- Denoting $K = -\beta J$, $B = -\beta h$, show that

$$Z = \text{tr}(T^L), \quad T = \begin{pmatrix} e^{K+B} & e^{-K} \\ e^{-K} & e^{K-B} \end{pmatrix},$$

where $T$ plays the role of a one-dimensional transfer matrix.

- Conclude that

$$Z = \Lambda_+^L + \Lambda_-^L, \quad \Lambda_\pm = e^K \cosh B \pm \sqrt{e^{2K} \sinh^2 B + e^{-2K}}.$$  

- Compute $\lim_{L \to \infty} \frac{\log Z}{L}$. Is there any phase transition in temperature, i.e., in the parameter $\beta$ (the inverse temperature) as it varies from 0 to $+\infty$?
2. Exercise session 2

2.1. The $q$-determinant. Consider the Yang–Baxter (bi)algebra associated to the six-vertex model $R$-matrix, with the standard notation $egin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ for the generating series of its generators.

- Prove the series of equalities
  
  $$q\det(z) := A(qz)D(z) - B(qz)C(z) = D(qz)A(z) - C(qz)B(z) = A(z)D(qz) - C(z)B(qz) = D(z)A(qz) - B(z)C(qz).$$

- Prove that $q\det(z)$ is central, i.e., commutes with all elements of the Yang–Baxter (bi)algebra, and prove that it is group like, i.e., $\Delta(q\det(z)) = q\det(z) \otimes q\det(z)$.

2.2. Commutation of twisted transfer matrices. Consider as above the Yang–Baxter (bi)algebra associated to the six-vertex model $R$-matrix. Recall that its defining relations, the RTT relations, can be written in components as sixteen relations for its generators $A(z), B(z), C(z), D(z)$.

- Write explicitly the components of the RTT relations involving $A(z)$ and $D(z)$ that you will need for the following part.
- Defining $T_\kappa(z) = A(z) + \kappa D(z), \quad \kappa \in \mathbb{C}$, conclude that $[T_\kappa(z), T_\kappa(z')] = 0$ for all $z, z'$.
  
  In other words, for a fixed $\kappa$, $T_\kappa(z)$ is the generating series for a commutative subalgebra of the Yang–Baxter algebra.

2.3. Bethe Ansatz equations as pole cancellations. Consider the six-vertex model with periodic boundary conditions, and its transfer matrix $T(z) = A(z) + D(z)$ acting on $(\mathbb{C}^2)^\otimes L$.

- Consider an eigenvector $|\Psi\rangle$ of $T(z)$, with eigenvalue
  
  $$T(z) |\Psi\rangle = t(z) |\Psi\rangle.$$ 

  As a function of $z$, what can be said about $t(z)$?
- Now assume $|\Psi\rangle$ is a Bethe vector. Write the formula expressing the eigenvalue $t(z)$ as a function of the Bethe roots $z_1, \ldots, z_M$. What is its dependence on $z$? Comparing with the previous part, conclude that the residues of $t(z)$ at the would-be poles of this formula must vanish. Compute these residues and compare with Bethe Ansatz equations.

2.4. Energy/momentum of XXZ eigenvectors.

- Using the trace identities, cf. exercise 1.3, compute the momentum and XXZ energy of a Bethe vector in terms of the Bethe roots. (Recall that the shift operator $U$ is unitary, so its eigenvalues are of the form $e^{ip}$ where $p \in \mathbb{R}/2\pi\mathbb{Z}$ is the momentum.)
- Argue that these states can be viewed as consisting of quasiparticles called magnons, where each magnon can be associated with one Bethe root.
- How does the isotropy at $\Delta = 1$ show up in the spectrum?
2.5. **Yang–Baxter algebra representations and inhomogeneous monodromy matrix.** We recall that a *representation* of an algebra $\mathcal{A}$ is the data of a vector space $V$ and an algebra morphism $\rho: \mathcal{A} \to \text{End}(V)$, i.e., a linear map preserving the multiplication. If $\mathcal{A}$ is a *bialgebra*, then one can take tensor products of representations using the coproduct $\Delta$: given $(V_1, \rho_1)$ and $(V_2, \rho_2)$, define the representation $(V_1 \otimes V_2, \rho_{1 \otimes 2})$ by $\rho_{1 \otimes 2}(a) = (\rho_1 \otimes \rho_2)\Delta(a)$ for all $a \in \mathcal{A}$.

Consider the Yang–Baxter bialgebra $\mathcal{A}$ with generators $(\hat{T}^j_i(z))_{i,j=1,\ldots,n}$ associated to an $R$-matrix $R(z) = (R_{ik}^{jl}(z))_{i,j,k,l=1,\ldots,n}$:

$$R_{ik}^{jl}(z/w) = \begin{array}{c|c}
  \ell & j \\
  k & w 
\end{array}$$

satisfying the Yang–Baxter equation. One may limit oneself to the case of the six-vertex model $R$-matrix, with $n = 2$: $\hat{T}_1^n(z) = \hat{A}(z)$, $\hat{T}_1^n(z) = \hat{B}(z)$, $\hat{T}_2^1(z) = \hat{C}(z)$, $\hat{T}_2^1(z) = \hat{D}(z)$.

- Show that $\hat{T}_i^j(z) \mapsto (R_{ik}^{jl}(z/w))_{k,l=1,\ldots,n}$ defines a representation of $\mathcal{A}$ on the vector space $\mathbb{C}^n$. This representation is often denoted $\mathbb{C}^n(w)$.
- Define the *inhomogeneous* monodromy matrix

$$T(z; z_1, \ldots, z_L) = R_{0L}(z/z_L) \cdots R_{01}(z/z_1) = \begin{array}{c|c|c|c}
  z & z_1 & z_2 & \cdots & z_L 
\end{array}$$

Note that the usual (homogeneous) monodromy matrix is the special case $T(z) = T(z; 1, \ldots, 1)$.

Show that $\hat{T}_i^j(z) \mapsto T_i^j(z; z_1, \ldots, z_L)$ is the tensor product representation $\mathbb{C}^n(z_1) \otimes \mathbb{C}^n(z_2) \otimes \cdots \otimes \mathbb{C}^n(z_L)$ of $\mathcal{A}$. 


3. Exercise session 3

3.1. Alternating Sign Matrices, tableaux and Gelfand–Tseytlin patterns. Recall that there is a bijection between six-vertex configurations with Domain Wall Boundary Conditions (DWBC) and Alternating Sign Matrices (ASMs).

- As a warm up find the DWBC configuration, both in the arrow picture and the path picture, and the domino tilings of the Aztec diamond corresponding to the ASM

$$\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{array}$$

The object of this exercise is to uncover more bijections. In the remainder one may use the following working example:

and provide the image of the example under the various bijections below.

A Gelfand–Tseytlin pattern is a triangle of (integer) numbers $(\lambda^{(i)}_j)_{1 \leq j \leq i, 1 \leq i \leq n}$ such that the inequalities hold $\lambda^{(i+1)}_j \geq \lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+1}$; i.e.,

$$\begin{array}{cccc}
\lambda^{(4)}_1 & \lambda^{(4)}_2 & \lambda^{(4)}_3 & \lambda^{(4)}_4 \\
\lambda^{(3)}_1 & \lambda^{(3)}_2 & \lambda^{(3)}_3 \\
\lambda^{(2)}_1 & \lambda^{(2)}_2 \\
\lambda^{(1)}_1 \\
\end{array}$$
A strict Gelfand–Tseytlin pattern (a.k.a. monotone triangle) is defined identically, except we further impose the strict inequality \( \lambda_{j+1}^{(i+1)} > \lambda_{j}^{(i+1)} \):

\[
\lambda_1^{(4)} > \lambda_2^{(4)} > \lambda_3^{(4)} > \lambda_4^{(4)}
\]

\[
\lambda_1^{(3)} > \lambda_2^{(3)} > \lambda_3^{(3)}
\]

\[
\lambda_1^{(2)} > \lambda_2^{(2)}
\]

\[
\lambda_1^{(1)}
\]

Finally, a Semi-Standard Young tableau (SSYT) is a filling \((T_{i,j})_{(i,j)\in Y}\) of a Young diagram \(Y\) with positive integers such \(T_{i,j} < T_{i+1,j}\) and \(T_{i,j} \leq T_{i,j+1}\), i.e.,

\[
\begin{array}{cccc}
1 & 2 & 3 & 3 \\
2 & 3 & 4 & \\
\end{array}
\]

We identify Young diagram with partitions – the example above is \((4,2,1)\).

- Given a DWBC configuration of size \(n\), define a triangular array of numbers as follows. The numbers are the columns of up-pointing arrows; more precisely, \(\lambda_j^{(i)}\) is the column number (counted left to right from 1 to \(n\)) of the \(j^{th}\) up-arrow (counted from the right) of the \(i^{th}\) row (counted from the bottom).

  Show that the resulting triangular array is a strict Gelfand–Tseytlin pattern, and that this provides a bijection between DWBC configurations of size \(n\) and strict Gelfand–Tseytlin patterns with top row \((n,...,2,1)\).

- Given a (not necessarily strict) Gelfand–Tseytlin pattern \((\lambda_j^{(i)})\) with nonnegative entries, one can produce a tableau as follows: each row \(\lambda_j^{(i)}\) of the pattern is a partition, which can be drawn as a Young diagram; we obtain this way a sequence of Young diagrams which is weakly decreasing w.r.t. inclusion. In turn, this gives a tableau of the partition of the top row \(\lambda^{(n)}\) as follows: a box of the Young diagram has label \(i \in \{1,...,n\}\) iff it belongs to (the Young diagram of) \(\lambda^{(i)}\) but not to \(\lambda^{(i-1)}\) (with the convention that \(\lambda^{(0)}\) is the empty partition).

  Show that the resulting tableau is semi-standard, and that this forms a bijection between Gelfand–Tseytlin patterns with fixed first row \(\lambda^{(n)}\) and the SSYTs of the partition \(\lambda^{(n)}\) with labels in \(\{1,...,n\}\).

- Given a DWBC configuration, apply successively the bijections of the two previous questions to produce a SSYT \((T_{i,j})\). What is its shape? Define a new triangular array by \(\lambda_j^{(i)}\) by \(\lambda_j^{(i)} = \lambda_{i+1-j,n+1-i} \). This corresponds to rotating and deforming the SSYT:

\[
\begin{array}{cccc}
T_{1,1} & T_{1,2} & T_{1,3} & T_{1,4} \\
T_{2,1} & T_{2,2} & T_{2,3} & T_{2,4} \\
T_{3,1} & T_{3,2} & T_{3,3} & T_{3,4} \\
T_{4,1} & T_{4,2} & T_{4,3} & \\
\end{array}
\]
Show that \( (\lambda^{(i)}) \) is again a strict Gelfand–Tseytlin pattern, and that it is associated via the first bijection to the \( \pi/2 \) clockwise rotation of the original DWBC configuration with all arrows reversed (or equivalently, to the \( \pi/2 \) clockwise rotation of the original ASM).

3.2. NilHecke solution of Yang–Baxter equation and Schubert polynomials. Let \( r \in \mathbb{Z}_{>0} \), and consider the following rational \( R \)-matrix:

\[
R^{ij}_{\ell k}(x - y) = \begin{cases} 
1 & i = \ell, \; k = j \\
(x - y) & i = j < k = \ell, \\
0 & \text{else}
\end{cases}, \quad 1 \leq i, j, k, \ell \leq r
\]

(where as usual, superscripts are row indices and subscripts are column indices). Viewed as an operator, it acts on \( \mathbb{C}^r \otimes \mathbb{C}^r \).

Given a \( n \)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \{1, \ldots, r\}^n \), we define \( \omega \) to be its “sort”, i.e., the only weakly increasing permutation of \( \lambda \). E.g., if \( \lambda = (2, 1, 3, 2) \), then \( \omega = (1, 2, 2, 3) \). We define the Schubert polynomial \( S_\lambda \) associated to \( \lambda \) to be the following partition function: (on this example, \( n = 4 \))

\[
S_\lambda = \lambda_1 x_1 \lambda_2 x_2 \lambda_3 x_3 \lambda_4 x_4 \prod_{i=1}^{n} (x_i - y_j)
\]

- Show that \( R \) satisfies the Yang–Baxter equation with additive spectral parameters.
- When \( r = 2 \), what is the \( R \)-matrix? Still at \( r = 2 \), what is \( S_\lambda \) when all \( y \)s are 0?
- For general \( r \), show that \( S_\lambda \) is a homogeneous polynomial in the \( x \)s and \( y \)s. What is its degree?
- Denote \( I_a = \{i : \omega_i = a\} \) for \( a = 1, \ldots, r \).
- Show that \( S_\lambda \) does not depend on the \( x_i, i \in I_r \).
- Show that for each \( a \in \{1, \ldots, r\} \), \( S_\lambda \) is invariant by permutation of the \( x_i, i \in I_a \).
- Given a \( n \)-tuple \( \lambda \), define its standardization \( \mu \) to be the unique \( n \)-tuple such that each integer in \( \{1, \ldots, n\} \) occurs once (i.e., \( \mu \) is a permutation) and for all \( i < j \) \( \lambda_i \leq \lambda_j \) iff \( \mu_i < \mu_j \). E.g., if \( \lambda = (2, 1, 3, 2) \) then \( \mu = (2, 1, 4, 3) \).
- Show that \( S_\lambda = S_\mu \) (where \( S_\mu \) is defined by choosing the value of \( r \) to be \( n \)).
- Define the inversion code \( \underline{\lambda} \) of a \( n \)-tuple \( \lambda \) to be the sequence

\[
\underline{\lambda} = (\#\{j > i : \lambda_j < \lambda_i\})_{i=1}^{n}
\]

e.g., if \( \lambda = (5, 2, 1, 3, 2) \) then \( \underline{\lambda} = (4, 1, 0, 1, 0) \). Show that if \( \underline{\lambda} \) is weakly decreasing, then

\[
S_\lambda = \prod_{j=1}^{n} (x_i - y_j)
\]

If \( r = 2 \), what are the \( \lambda \) satisfying this condition?