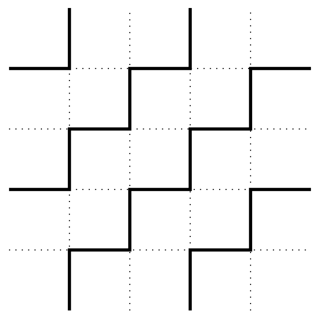
QUANTUM INTEGRABILITY AND SYMMETRIC POLYNOMIALS

Solutions

1.1. Low-temperature expansion of the antiferroelectric six-vertex model.

- *Describe the two ground states of the model, i.e., the configurations of the model with maximal Boltzmann weight.*

  The ground states involve only vertices with weight c. In the path picture the ground states look like staircases: one is

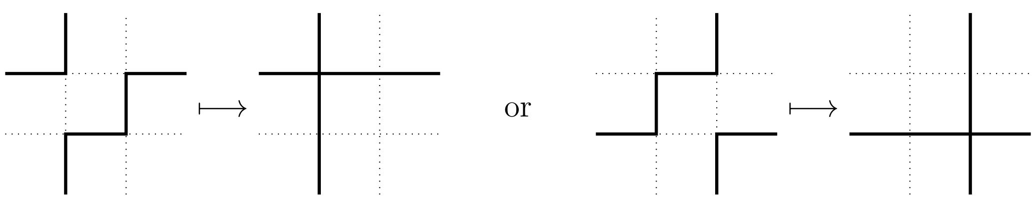
  

  and the other its parity reverse, which is also obtained by translation of the preceding by one in any direction. (Note that the above is only compatible with periodic boundary conditions when K and L are even.)

- *Use graphical notations to compute the partition function to ninth order in a/c, b/c.*

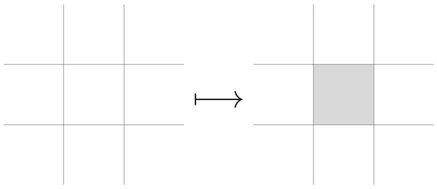
  By the previous part and parity symmetry we can focus on the 'sector' of configurations close to one of the two ground states, and multiply the result by 2 to get Z. Thus we compute $Z/2$. If we rescale the weights such that $c = 1$ the ground state in this sector has weight 1, so $\frac{1}{2}Z = 1 + h.d.$, where '*h.d.*' indicates subleading terms, of higher (total) degree in a, b.

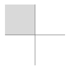
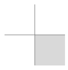
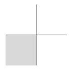
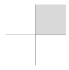
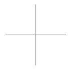
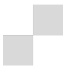
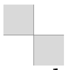
  The smallest change that the ice rule allows us to make is to pick any face of the lattice and reverse parity on the edges bounding that face:

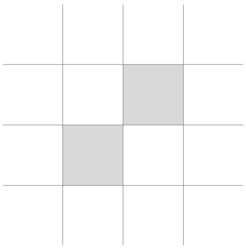
  In either case the resulting weight is a^2b^2. We can choose any of the $V = KL$ faces, so $\frac{1}{2}Z = 1 + V\, a^2b^2 + h.d.$

  Before we go on we notice that the graphical notation may be simplified for the purpose of this exercise. Indeed, it is somewhat cumbersome to have to distinguish the two cases as above, depending on which face we choose precisely. Both of the above, however, resulted in the same weight: this is a consequence of the fact, cf. the previous part, that for the ground state, translation by one in any direction is equivalent to reversing all spins. Let us therefore only indicate the face around which we reversed the spins, so that *both* of the preceding look like

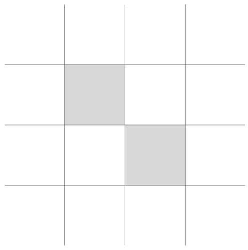


Moreover, it is easy to read off the weights:  $=$  $= a$,  $=$  $= b$,  $=$  $=$  $= c = 1$, while any configuration around a vertex such that two gray rectangles share an edges has weight zero. Importantly, one should convince oneself that the vertex weights can be read off from the grey-face configurations regardless of the where the thick and thin zigzag lines from the ground-state configuration run. This will simplify the combinatorics, and we're all set for a nice graphical perturbation theory to compute the partition function.

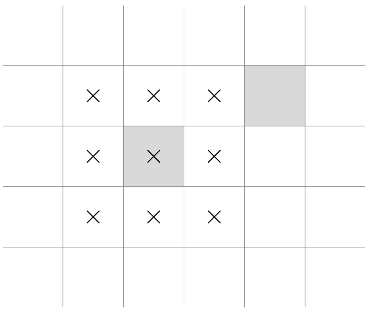
Let's see what happens when we colour *two* faces of the lattice in grey. Recalling that the two grey faces cannot share an edge there are two possibilities. Firstly the two grey squares can share one vertex:



or

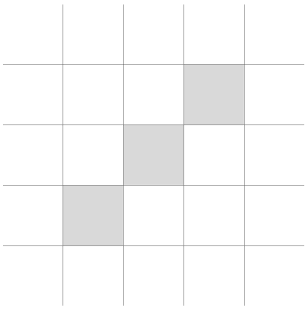
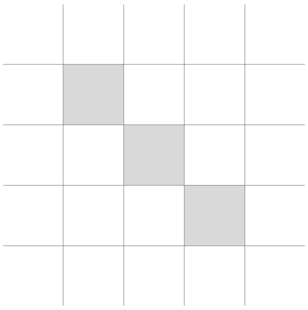
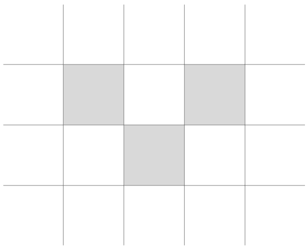


The left has weight a^4b^2 and the right has weight a^2b^4; clearly either shape can be put on V different locations on the lattice. The second possibility is when the two grey faces have no vertices in common:



The weight is $(a^2b^2)^2$, and there are $\frac{1}{2}V(V-9)$ possibilities: choosing the first grey face excludes the faces marked by $\times$ for the second grey face, and the order in which we choose the faces does not matter so we have to include a factor of $\frac{1}{2}$ to avoid overcounting. So far we have $\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V-9)a^4b^4 + \cdots$, though at this point it's not clear yet what the degree of the next terms in the expansion will be.

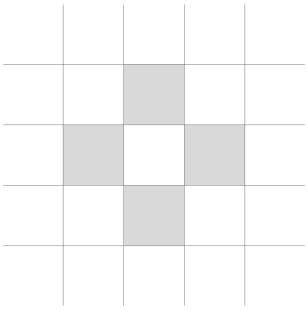
Next we consider *three* grey faces. When at least one of these faces has no vertex in common with either of the other two then the weight is a^2b^2 times what we got for two grey faces, yielding total degree ≥ 10 in a, b, which is further than we want to go. This leaves the following possibilities for us to consider:

+ 3 rotations over 90°

The first has weight a^6b^2, the second a^2b^6 and the third a^4b^4 (by symmetry in a, b each rotation over 90° of the latter has the same weight). Clearly there are V ways to place each of these, so $4V$ of the latter if we include the rotated versions. Our tally thus is $\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V-1)a^4b^4 + Va^2b^2(a^4+b^4) + \cdots$.

Next we consider *four* grey faces. The lowest possible total degree is attained when the grey faces share as many vertices as possible but have no edges in common:

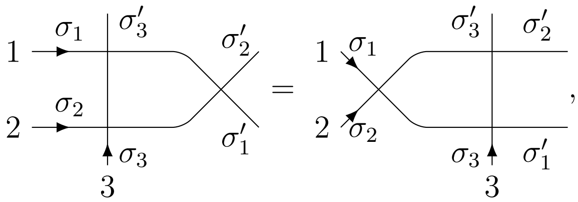


which has weight a^4b^4, and V ways to be placed on the lattice. Observe that any other way to distribute at least four grey faces will yield higher total degree. We therefore arrive at $\frac{1}{2}Z = 1 + Va^2b^2 + Va^2b^2(a^2+b^2) + \frac{1}{2}V(V+1)a^4b^4 + Va^2b^2(a^4+b^4) + h.d.$, where '*h.d.*' has total degree ≥ 10 in a, b. This is what we wanted to show.

- *Explain the symmetry of Z in a, b from the symmetries of the model.*

  As is clear in the arrow picture or the grey-face picture a rotation over 90° exchanges $a \leftrightarrow b$ and preserves c. (In particular, a rotation over 180° preserves all weights, cf. the second part of the next exercise.) In the above we saw that contributions are either invariant under such rotations, or come in pairs related by such a rotation.

1.2. Determination of the six-vertex *R*-matrix from *RLL* relations.



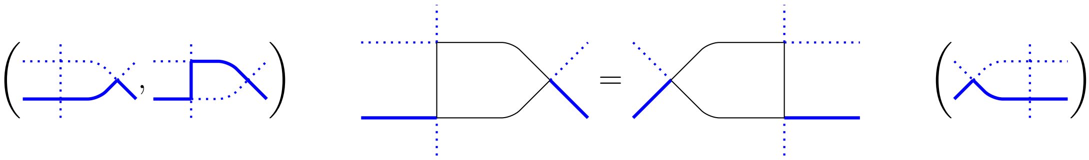
$\sigma_1, \sigma_2, \sigma_3, \sigma_1', \sigma_2', \sigma_3' \in \{\pm 1\}$

- *Show that due to line conservation out of these $2^6 = 64$ equations only $\sum_{k=0}^{3} \binom{3}{k}^2 = 20$ are nontrivial, which due to parity symmetry come in pairs of identical equations.*

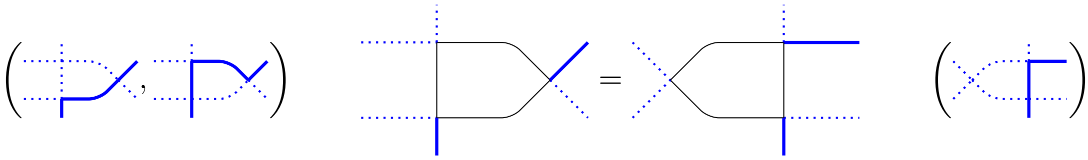
  This is a consequence of the ice rule: at each vertex we must have $\sum_{\text{incoming}} \sigma = \sum_{\text{outgoing}} \sigma$ for the Boltzmann weight to be nonzero. By summing over the three vertices, we find that both l.h.s. and r.h.s. are zero unless $\sum_{i=1}^{3} \sigma_i = \sum_{i=1}^{3} \sigma'_i$. If $k = \#\{i : \sigma_i = +1\}$ this gives the desired counting.

- *Noting that the left and right hand sides are exchanged by 180° rotation of the corresponding pictures, conclude that out of the ten equations, four correspond to 180° rotation invariant boundary conditions and are therefore automatically satisfied, while six come in pairs of identical equations and read:*

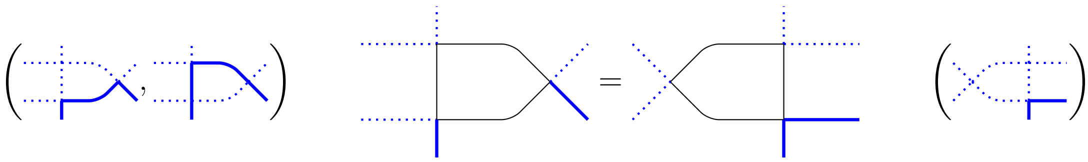
$$ab'c'' + cc'b'' = ba'c'',$$



$$ac'b'' + cb'c'' = bc'a'',$$



$$ac'c'' + cb'b'' = ca'a''.$$



We have indicated below each equation the corresponding choice of boundary conditions, as well as the actual configurations.

- *Eliminating a'', b'', c'', show that a solution for R exists only if $\Delta(a, b, c) = \Delta(a', b', c')$ with*

$$\Delta(a, b, c) = \frac{a^2 + b^2 - c^2}{2ab}.$$

a'', b'', c'' satisfy the following set of linear equations

$$\begin{pmatrix} 0 & cc' & ab' - ba' \\ -bc' & ac' & cb' \\ -ca' & cb' & ac' \end{pmatrix} \begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} = 0$$

For R to be invertible, one must have in particular $(a'', b'', c'') \neq (0, 0, 0)$, which means the linear system above is degenerate. Computing the determinant of the matrix yields

$$\begin{vmatrix} 0 & cc' & ab' - ba' \\ -bc' & ac' & cb' \\ -ca' & cb' & ac' \end{vmatrix} = cc'(a'b'(a^2 + b^2 - c^2) - ab(a'^2 + b'^2 - c'^2))$$
$$= 2aa'bb'cc'(\Delta(a, b, c) - \Delta(a', b', c'))$$

Recalling that we have assumed $a, b, c, a', b', c' > 0$, we conclude that $\Delta(a, b, c) = \Delta(a', b', c')$.

- *Show that if $\Delta(a, b, c) = \Delta(a', b', c')$ then one also has $\Delta(a'', b'', c'') = \Delta(a, b, c)$, i.e., the R-matrix is of the same form as the L-matrices.*

Note that the first two equations are swapped by exchanging primes and double primes, whereas the third equation is left invariant. We can therefore apply the same reasoning as above to conclude that $\Delta(a,b,c) = \Delta(a'',b'',c'')$.

- *Show that if L is parameterized as*

$$\begin{aligned} a(z) &= q\,z - q^{-1}z^{-1}, \\ b(z) &= z - z^{-1}, \\ c(z) &= q - q^{-1}, \end{aligned}$$

and L' similarly with z replaced by z', then R is of the same form with spectral parameter $z'' = z/z'$.

Any row of the comatrix of the degenerate linear system above solves it. For example taking the last two rows leads to

$$\begin{aligned} \begin{pmatrix} a'' \\ b'' \\ c'' \end{pmatrix} \propto \begin{pmatrix} a^2c'^2 - b'^2c^2 \\ abc'^2 - a'b'c^2 \\ -bb'cc' + aa'cc' \end{pmatrix} &= (q-q^{-1})^2 \begin{pmatrix} a^2 - b'^2 \\ ab - a'b' \\ -bb' + aa' \end{pmatrix} \\ &= (q-q^{-1})^2 (qzz' - (qzz')^{-1}) \begin{pmatrix} q\,z/z' - q^{-1}z'/z \\ z/z' - z'/z \\ q - q^{-1} \end{pmatrix} \end{aligned}$$

1.3. **XXZ Hamiltonians.**

- *Show that up to normalization $T(1)$ is the translation operator U, i.e., $U\, e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_L} = e_{k_L} \otimes e_{k_1} \otimes \cdots \otimes e_{k_{L-1}}$ where e_+, e_- are standard basis vectors.*

  First note that $R(1) = (q-q^{-1})\,P$ with P the permutation on $\mathbb{C}^2 \otimes \mathbb{C}^2$, i.e. $R_{0i} = (q-q^{-1})\,P_{0i}$ in the notation $P_{0i}\, e_{k_0} \otimes e_{k_1} \cdots \otimes e_{k_i} \otimes \cdots \otimes e_{k_L} = e_{k_i} \otimes e_{k_1} \cdots \otimes e_{k_0} \otimes \cdots \otimes e_{k_L}$. Therefore $T(1) = (q-q^{-1})^L\, \mathrm{tr}_0[P_{0L} \cdots P_{02}P_{01}]$. To compute the partial trace we note that, if P_π denotes the permutation on $(\mathbb{C}^2)^{\otimes(L+1)}$ corresponding to $\pi \in S_{L+1}$ so that $P_{(0i)} = P_{0i}$, then $P_{0L} \cdots P_{02}P_{01} = P_{(012\cdots L)} = P_{01}\, P_{(12\cdots L)}$. Since moreover $\mathrm{tr}_0 P_{0i} = \mathrm{id}$ we conclude that $T(1) = (q-q^{-1})^L\, \mathrm{tr}_0[P_{01}]\, P_{(12\cdots L)}$, with $P_{(12\cdots L)} = U$ the right shift. (All of these identities can be verified by acting on a basis vector $e_{k_1} \otimes \cdots \otimes e_{k_L}$.)
- *Show that up to an additive constant and normalization $(\log T)'(1) = \frac{dT}{dz}T^{-1}|_{z=1}$ is the XXZ Hamiltonian.*

Since the transfer matrix forms a one-parameter family of commuting operators we have $(\log T)'(1) = T^{-1}\frac{dT}{dz}|_{z=1} = \frac{dT}{dz}T^{-1}|_{z=1}$. We compute

$$\begin{aligned}
(q-q^{-1})^{1-L}\,\frac{dT}{dz}\Big|_{z=1} &= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,R'_{0i}(1)\,P_{0\,i-1}\cdots P_{02}P_{01}] \\
&= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,P_{0i}\,\check{R}'_{0i}(1)\,P_{0\,i-1}\cdots P_{02}P_{01}] \\
&= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,\check{R}'_{i\,0}(1)\,P_{0i}\,P_{0\,i-1}\cdots P_{02}P_{01}] \\
&= \sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots \check{R}'_{i\,i+1}(1)P_{0\,i+1}\,P_{0i}\,P_{0\,i-1}\cdots P_{02}P_{01}] \\
&= \sum_{i=1}^{L} \check{R}'_{i\,i+1}(1)\,\mathrm{tr}_0[P_{0L}\cdots P_{0\,i+1}\,P_{0i}\,P_{0\,i-1}\cdots P_{02}P_{01}].
\end{aligned}$$

Recognize the shift operator from above to get $(\log T)'(1) = (q-q^{-1})^{-1}\sum_{i=1}^{L}\check{R}'_{i\,i+1}(1)$. As

$$\check{R}'_{i\,i+1}(1) = \begin{pmatrix} 2\Delta & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2\Delta \end{pmatrix} = \big(2\,(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+) + \Delta\,(\sigma_i^z\sigma_{i+1}^z + 1)\big)$$

we conclude that $(\log T)'(1) = (q-q^{-1})^{-1}\,(H_{\mathrm{XXZ}} + L\,\Delta)$.

- (*Optional*) *Compute the second XXZ Hamiltonian* $(\log T)''(1)$.

  Note that $(\log T)''(1) = (T'\,T^{-1})'(1) = T''(1)\,T(1)^{-1} - (T'(1)\,T(1)^{-1})^2$. Proceeding like before we get

$$\begin{aligned}
T''(1) = (q-q^{-1})^{L-1} &\sum_{i=1}^{L} \mathrm{tr}_0[P_{0L}\cdots P_{0i}\,\check{R}''_{0i}(1)\,P_{0\,i-1}\cdots P_{02}P_{01}] \\
&+ 2\,(q-q^{-1})^{L-2}\sum_{i<j} \mathrm{tr}_0[P_{0L}\cdots P_{0i}\,\check{R}'_{0i}(1)\,P_{0\,i-1}\cdots \\
&\qquad\qquad \cdots P_{0j}\,\check{R}'_{0j}(1)\,P_{0\,j-1}\cdots P_{02}P_{01}],
\end{aligned}$$

where we used $\sum_{i<j}\cdots + \sum_{j>i}\cdots = 2\sum_{i<j}\cdots$. Therefore

$$\begin{aligned}
T''(1)T(1)^{-1} = (q-q^{-1})^{-1}\sum_{i=1}^{L}\check{R}''_{i\,i+1}(1) &+ 2\,(q-q^{-1})^{-2}\sum_{i<j-1}^{L}\check{R}'_{i\,i+1}(1)\,\check{R}'_{j\,j+1}(1) \\
&+ 2\,(q-q^{-1})^{-2}\sum_{i=1}^{L}\check{R}'_{i\,i+1}(1)\,\check{R}'_{i+1\,i+2}(1).
\end{aligned}$$

On the other hand, by the previous part

$$(T'(1)T(1)^{-1})^2 = (q-q^{-1})^{-2}\Big(2\sum_{i<j-1}^{L}\check R'_{i\,i+1}(1)\,\check R'_{j\,j+1}(1) + \sum_{i=1}^{L}\check R'_{i\,i+1}(1)\check R'_{i+1\,i+2}(1)$$
$$+\sum_{i=1}^{L}\check R'_{i\,i+1}(1)^2 + \sum_{i=1}^{L}\check R'_{i\,i+1}(1)\check R'_{i-1\,i}(1)\Big).$$

The double sums (with terms with $|i-j|>2$) cancel, so obtain

$$(\log T)''(1) = (q-q^{-1})^{-1}\sum_{i=1}^{L}\big(\check R''_{i\,i+1}(1) - \check R'_{i\,i+1}(1)^2\big)$$
$$+(q-q^{-1})^{-2}\sum_{i=1}^{L}\big[\check R'_{i\,i+1}(1), \check R'_{i+1\,i+2}(1)\big]$$

where we recognised a commutator that acts nontrivially at *three* neighbouring sites. (This pattern continues, and the higher Hamiltonians computed in this way will be less and less local. A better set of conserved charges that are *quasilocal* can be obtained from transfer matrices with *higher-spin* auxiliary spaces, by fusion, but this is beyond the scope of this course.)

1.4. **The 1D Ising model.**

- *How is H_{Ising} related to H_{XXZ} of the previous exercise?*

  We have: $H_{\text{Ising}} = J\lim_{\Delta\to\infty}\Delta^{-1}H_{\text{XXZ}} + h\sum_{i=1}^{L}\sigma_i^z$. Because $[H_{\text{XXZ}}, \sum_{i=1}^{L}\sigma_i^z]=0$, diagonalization of H_{Ising} is equivalent to that of $\lim_{\Delta\to\infty}\Delta^{-1}H_{\text{XXZ}}$. Furthermore, if one takes $h=0$ and J and Δ are of the same sign, the two models (XXZ at $\Delta\to\pm\infty$ and Ising) are therefore equivalent.
- *Show that $Z = tr(e^{-\beta H_{Ising}})$ is the partition function of the classical one-dimensional Ising model.*

  H_{Ising} is diagonal in the standard basis $\bigotimes_{i=1}^{L} e_{\sigma_i}$ of $(\mathbb{C}^2)^{\otimes L}$, where $\sigma\in\{\pm\}^L$, with corresponding eigenvalue

  $$E_{\text{Ising}}(\sigma) = \sum_{i=1}^{L}\sigma_i\sigma_{i+1} - \beta h\sum_{i=1}^{L}\sigma_i$$

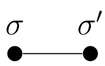
  Therefore $\text{tr}(e^{-\beta H_{\text{Ising}}})$ is the sum of $e^{-\beta\text{ eigenvalues}}$, which is the desired expression.
- *Denoting $K=-\beta J$, $B=-\beta h$, show that*

  $$Z = tr(T^L),\qquad T = \begin{pmatrix} e^{K+B} & e^{-K}\\ e^{-K} & e^{K-B}\end{pmatrix},$$

  We use the transfer matrix approach (this is a spin model, but in 1D, vertex models and spin models are actually equivalent by exchanging edges and vertices). We first split the energy in terms of edges only as

  $$E_{\text{Ising}}(\sigma) = J\sum_{i=1}^{L}(\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i+\sigma_{i+1})).$$

  and then consider the matrix encoding a single edge:

  $T =$  $= (e^{-\beta J\sigma\sigma' - \beta\frac{h}{2}(\sigma+\sigma')})_{\sigma,\sigma'\in\{\pm\}}$

Concatenating successive edges corresponds to taking powers of T, and the periodic boundary conditions correspond to taking the trace, so that we find $Z = \mathrm{tr}(T^L)$ as expected.

- *Conclude that*

$$Z = \Lambda_+^L + \Lambda_-^L, \qquad \Lambda_\pm = e^K \cosh B \pm \sqrt{e^{2K} \sinh^2 B + e^{-2K}}.$$

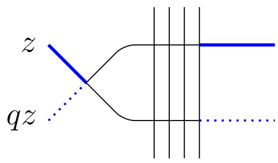
  $\Lambda_\pm$ are the eigenvalues of T.

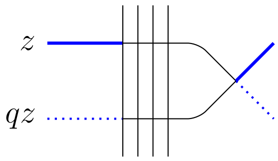
- *Compute* $\lim_{L\to\infty} \frac{\log Z}{L}$. *Is there any phase transition in temperature, i.e., in the parameter* β *(the inverse temperature) as it varies from* 0 *to* $+\infty$*?*

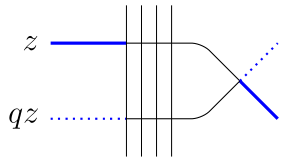
  Note $\Lambda_+ > \Lambda_- > 0$. So $\lim_{L\to\infty} \frac{\log Z}{L} = \log \Lambda_+$. This function is analytic in β, so no phase transition occurs.

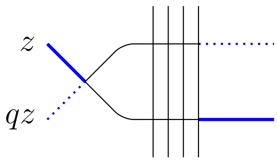
2.1. The q-determinant.

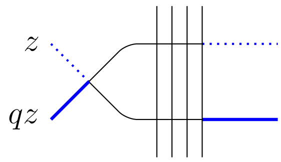
- *Prove the series of equalities for* $\mathrm{qdet}(z)$. Note that if we specialize the spectral parameters in the RTT relations to $z_1 = z$ and $z_2 = qz$ then the R-matrix has argument $z_1/z_2 = q^{-1}$, at which $a(q^{-1}) = 0$ while $c(q^{-1}) = q - q^{-1} = -b(q^{-1})$. Thus the nontrivial equations contained in the RTT relations in this case gives the following series of equalities:

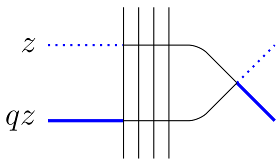
$(q - q^{-1})\,\mathrm{qdet}(z) = (q - q^{-1})\big(A(qz)D(z) - B(qz)C(z)\big) =$ 

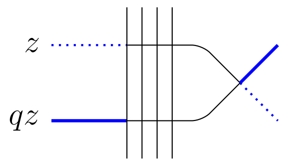
$\overset{RTT}{=}$ 

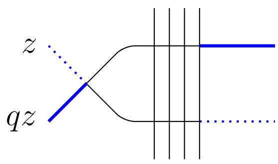
$= (q - q^{-1})\big(A(z)D(qz) - C(z)B(qz)\big) = -1 \times$ 

$\overset{RTT}{=} -1 \times$ 

$= (q - q^{-1})\big(D(qz)A(z) - C(qz)B(z)\big) =$ 

$\overset{RTT}{=}$ 

$= (q - q^{-1})\big(D(z)A(qz) - B(z)C(qz)\big) = -1 \times$ 

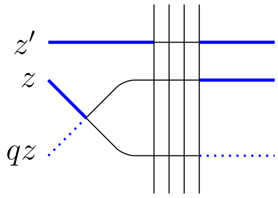
$\overset{RTT}{=} -1 \times$ 

$= (q - q^{-1})\big(A(qz)D(z) - B(qz)C(z)\big) = (q - q^{-1})\,\mathrm{qdet}(z).$

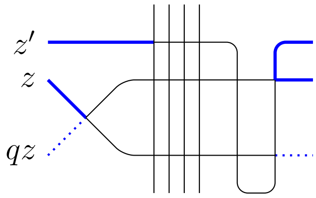
- *Prove that* $\mathrm{qdet}(z)$ *is central and group-like.*

For centrality one needs to prove that $\mathrm{qdet}(z)$ commutes with $A(z')$, $B(z')$, $C(z')$, $D(z')$. Let us consider $A(z')$

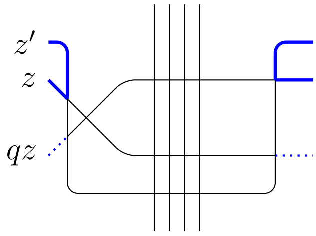
$$(q-q^{-1})A(z')\mathrm{qdet}(z) =$$



$$\overset{\text{unitarity}}{=} \frac{1}{a(z/z')a(z'/z)a(q\,z/z')a(q^{-1}z'/z)}$$

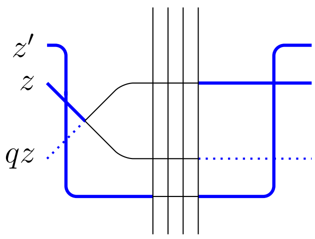


$$\overset{\text{RTT,YBE}}{=} \frac{1}{a(z/z')a(z'/z)a(q\,z/z')a(q^{-1}z'/z)}$$

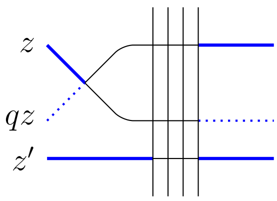


The ice rule and the fact that $a(q^{-1}) = 0$ fixes various internal edges:

$$= \frac{1}{a(z/z')a(z'/z)a(q\,z/z')a(q^{-1}z'/z)}$$



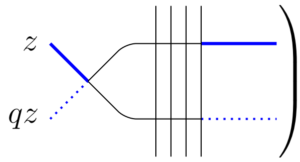
$$= \frac{a(z'/z)b(z'/(qz))b(z/z')a(q\,z/z')}{a(z/z')a(z'/z)a(q\,z/z')a(q^{-1}z'/z)}$$



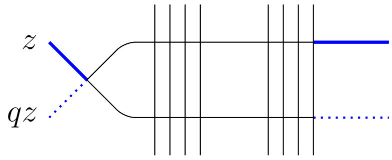
$$= (q-q^{-1})\mathrm{qdet}(z)A(z')$$

The other commutations are proved similarly.

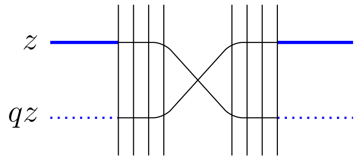
$$\Delta(\mathrm{qdet}(z)) = \Delta(A(qz)D(z) - B(qz)C(z)) = \frac{1}{q-q^{-1}}\,\Delta$$



$$= \frac{1}{q-q^{-1}}$$



$$\overset{RTT}{=} \frac{1}{q-q^{-1}}$$



Summing over the four possibilities for the middle vertex (recall that a vanishes for our choice of spectral parameters), we immediately obtain

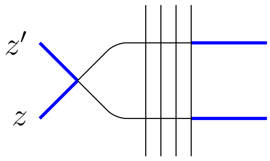
$$\begin{aligned}\Delta(\mathrm{qdet}(z)) &= A(z)D(qz) \otimes A(qz)D(z) - A(z)D(qz) \otimes B(qz)C(z) \\ &\quad - C(z)B(qz) \otimes A(qz)D(z) + C(z)B(qz) \otimes B(qz)C(z) \\ &= (A(z)D(qz) - C(z)B(qz)) \otimes (A(qz)D(z) - B(qz)C(z)) \\ &= \mathrm{qdet}(z) \otimes \mathrm{qdet}(z)\end{aligned}$$

2.2. **Commutation of twisted transfer matrices.**

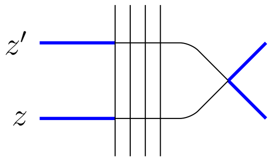
- *Write explicitly the components of the RTT relations involving $A(z)$ and $D(z)$ that you will need for the following part.*

  We will need to commute $A(z)$ and $D(z)$ past $A(z')$ and $D(z')$. The pertinent relations are encoded in the RTT relations. For $A(z)$ and $A(z')$ we have

$$a(z'/z)A(z)A(z') =$$



$$\overset{RTT}{=}$$

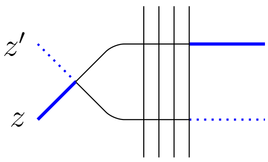


$$= a(z'/z)A(z')A(z)$$

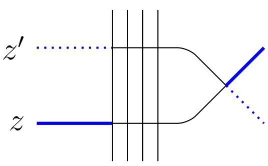
where the outer equalities use the ice rule. So $[A(z), A(z')] = 0$ for generic spectral parameters (i.e., except when $a(z/z') = 0$, cf. the previous exercise). By reversing the spin on all horizontal external lines we likewise obtain $[D(z), D(z')] = 0$.

For the remaining commutation relation we use

$$b(z'/z)A(z)D(z') + c(z'/z)B(z)C(z') =$$



$$\overset{RTT}{=}$$



$$= b(z'/z)D(z')A(z) + c(z'/z)B(z')C(z)$$

whence $[A(z), D(z')] = c(z'/z)/b(z'/z)\,(B(z')C(z) - B(z)C(z'))$. The appearance of B and C is not a problem, and we won't need any of the further relations between A and D, as we will show next.

- *Defining $T_\kappa(z) = A(z) + \kappa D(z)$, $\kappa \in \mathbb{C}$, conclude that $[T_\kappa(z), T_\kappa(z')] = 0$ for all z, z'.*
  Since $b(z) = -b(z^{-1})$ we have

$$\begin{aligned}
[A(z), D(z')] &= \frac{c(z'/z)}{b(z'/z)}\,(B(z')C(z) - B(z)C(z')) \\
&= -\frac{c(z/z')}{b(z/z')}\,(B(z')C(z) - B(z)C(z')) \\
&= -\frac{c(z'/z)}{b(z'/z)}\,(-B(z)C(z') + B(z')C(z)) \\
&= \frac{c(z'/z)}{b(z'/z)}\,(B(z)C(z') - B(z')C(z)) \\
&= [A(z'), D(z)] = -[D(z), A(z')].
\end{aligned}$$

This shows that we get cancellation at each other of κ separately:

$$[T_\kappa(z), T_\kappa(z')] = [A(z), A(z')] + \kappa\big([A(z), D(z')] + [D(z), A(z')]\big) + \kappa^2[D(z), D(z')] = 0.$$

2.3. Bethe Ansatz equations as pole cancellations.

- *Consider an eigenvector $|\Psi\rangle$ of $T(z)$, with eigenvalue $T(z)\,|\Psi\rangle = t(z)\,|\Psi\rangle$. As a function of z, what can be said about $t(z)$?*
  Since $T(z)$ is a polynomial in the vertex weights, and the latter are Laurent polynomials in z, so are the entries of $T(z)$. Therefore the eigenvalues $t(z)$ must be Laurent polynomials in the spectral parameter too. In particular it can only have poles at $z = 0, \infty$.
- *Now assume $|\Psi\rangle$ is a Bethe vector. Comparing with the previous part, conclude that the residues of $t(z)$ at the would-be poles of this formula must vanish. Compute these residues and compare with Bethe Ansatz equations.*
  Recall that

$$t(z) = A^{(0)}(z)\prod_{k=1}^{M}\frac{a(z_k/z)}{b(z_k/z)} + D^{(0)}(z)\prod_{k=1}^{M}\frac{a(z/z_k)}{b(z/z_k)}, \qquad \begin{aligned} A^{(0)}(z) &= a(z)^L, \\ D^{(0)}(z) &= b(z)^L. \end{aligned}$$

Due to the denominators this expression has (simple) poles at $z = z_j$, $1 \le j \le M$ besides those at $z = 0, \infty$. Since $b(z) = -b(z^{-1})$ the residue at these poles can be computed as

$$\begin{aligned}
\mathrm{Res}_{z=z_j} t(z) &= \mathrm{Res}_{z=z_j}\frac{A^{(0)}(z)\prod_{k=1}^{M} a(z_k/z) + (-1)^M D^{(0)}(z)\prod_{k=1}^{M} a(z/z_k)}{\prod_{k=1}^{M} b(z/z_k)} \\
&= \frac{A^{(0)}(z_j)\prod_{k=1}^{M} a(z_k/z_j) + (-1)^M D^{(0)}(z)\prod_{k=1}^{M} a(z_j/z_k)}{b'(1)/z_k\prod_{k(\neq j)}^{M} b(z_j/z_k)}.
\end{aligned}$$

The condition that the numerator vanishes is precisely the Bethe ansatz equations, which therefore ensure that the eigenvalues of the Bethe vectors have the correct analytic properties in z.

2.4. **Energy/momentum of XXZ eigenvectors.**

- *Compute the momentum and XXZ energy of a Bethe vector.*

  From the algebraic Bethe ansatz we know that, in the sector with M excitations, the transfer matrix has eigenvalues

$$t(z) = A^{(0)}(z) \prod_{k=1}^{M} \frac{a(z_k/z)}{b(z_k/z)} + D^{(0)}(z) \prod_{k=1}^{M} \frac{a(z/z_k)}{b(z/z_k)}, \qquad A^{(0)}(z) = a(z)^L, \quad D^{(0)}(z) = b(z)^L,$$

provided $z_1, \cdots, z_M$ obey the Bethe ansatz equations for $1 \leq j \leq M$:

$$A^{(0)}(z_j) \prod_{k=1}^{M} a(z_k/z_j) + (-1)^M D^{(0)}(z_j) \prod_{k=1}^{M} a(z_j/z_k) = 0.$$

In the first tutorial we found the 'trace identities' $T(1) = (q-q^{-1})^L\, U$ and $(\log T)'(1) = (q-q^{-1})^{-1}\,(H_{\mathrm{XXZ}} + L\,\Delta)$. It follows that the momentum and energy can be found from the same expressions for the eigenvalues. As $D^{(0)}(1) = 0$ we find

$$(q-q^{-1})^L\, e^{i\,p} = t(1) = a(1)^L \prod_{k=1}^{M} \frac{a(z_k)}{b(z_k)} + 0 = (q-q^{-1})^L \prod_{k=1}^{M} \frac{q z_k - q^{-1} z_k^{-1}}{z_k - z_k^{-1}},$$

so the momentum is

$$p = -i \sum_{k=1}^{M} \log \frac{q z_k - q^{-1} z_k^{-1}}{z_k - z_k^{-1}}.$$

Since $D^{(0)\prime}(1) = 0$ too (as long as $L > 1$)

$$\begin{aligned} t'(1) &= t(1) \Big(L\, \frac{a'(1)}{a(1)} + \sum_{k=1}^{M} (-z_k) \Big(\frac{a'(z_k)}{a(z_k)} - \frac{b'(z_k)}{b(z_k)} \Big) + 0 \Big) \\ &= t(1) \Big(L\, \frac{2\,\Delta}{q-q^{-1}} + \sum_{k=1}^{M} \frac{2\,(q-q^{-1})}{(q z_k - q^{-1} z_k^{-1})(z_k - z_k^{-1})} \Big). \end{aligned}$$

From the trace identity

$$(q-q^{-1})^{-1}\,(E_{\mathrm{XXZ}} + L\,\Delta) = (\log t)'(1) = t'(1)/t(1)$$

we thus get

$$E_{\mathrm{XXZ}} = L\,\Delta + \sum_{k=1}^{M} \frac{2\,(q-q^{-1})^2}{(q z_k - q^{-1} z_k^{-1})(z_k - z_k^{-1})}.$$

- *Argue that these states can be viewed as consisting of quasiparticles called magnons, where each magnon can be associated with one Bethe root.*

  Note that $E^{(0)}_{\mathrm{XXZ}} = L\,\Delta$ is just the energy of $|{+}+\cdots+\rangle$, which has momentum $p = 0$ (by translational invariance). Therefore both the momentum and energy of the Bethe states compared to the reference ('pseudovacuum') vector $|{+}+\cdots+\rangle$ has a nice additive form:

$$p = \sum_{k=1}^{M} p(z_k), \qquad p(z) = -i \log \frac{q z - q^{-1} z^{-1}}{z - z^{-1}},$$

$$E_{\mathrm{XXZ}} - E^{(0)}_{\mathrm{XXZ}} = \sum_{k=1}^{M} \varepsilon(z_k), \qquad \varepsilon(z) = \frac{2\,(q-q^{-1})^2}{(q z - q^{-1} z^{-1})(z - z^{-1})}.$$

The physical interpretation is that each $B(z_k)$ in the algebraic Bethe ansatz creates an excitation (quasiparticle), called a magnon, which contributes 'quasimomentum' $p(z_k)$ to the momentum, and 'quasienergy' $\varepsilon(z_k)$ to the energy of the Bethe vector. For $M = 1$ this magnon is a single $-$ travelling in a 'sea' of $+$s. For $M > 1$ the interactions between the particles, leading to bound states and so on, are accounted for by the Bethe equations coupling the z_k. Note that, unless $\Delta < -1$, the reference vector $|+\cdots+\rangle$ is *not* the ground state of the model and the magnons are *not* the physical low-lying excitations.

- *How does the isotropy at $\Delta = 1$ show up in the spectrum?*

Naively the preceding gives $p(0)|_{q=1} = \varepsilon(0)|_{q=1} = 0$ in the isotropic limit $q \to 1$, but this cannot be right as the resulting XXX Hamiltonian is nontrivial. The point is that the parametrization of the vertex weights that we used is not well suited for this limit; c vanishes in this limit. One way to proceed is to reparametrize the vertex weights to ensure that the limit $q \to 1$ remains meaningful. We will get back to this momentarily. For now the issue can be circumvented by expressing ε as a function of the quasimomentum: solving $p(z_k) = p_k$ for z_k^2 and substituting the result in ε gives $\varepsilon(p_k) = 4(\cos p_k - \Delta)$. At $\Delta = 1$ this 'dispersion relation' remains nontrivial, and may be rewritten in the form $\varepsilon_{\Delta=1}(p_k) = 2\sin^2(p_k/2)$. We see that, in this case, magnons with $p_k = 0$ contribute $\varepsilon_{\Delta=1}(0) = 0$ to the energy. This is compatible with the isotropy (i.e., full $\mathfrak{sl}_2$ symmetry) of the XXX Hamiltonian, which leads to degeneracies in the spectrum (higher-dimensional eigenspaces).

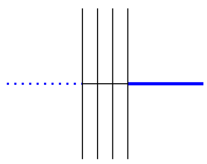
In fact, we claim that for $\Delta = 1$ creating a magnon with $p_k = 0$ corresponds to acting with the $\mathfrak{sl}_2$-lowering operator $\Sigma^- = \sum_{i=1}^{L} \sigma_i^-$ on $(\mathbb{C}^2)^{\otimes L}$. To see this we rescale the vertex weights to set c to unity and write $z = q^x$ before letting $q \to 1$:

$$\lim_{q\to 1} \frac{L(q^x)}{q-q^{-1}} = \begin{pmatrix} x+1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 0 & x+1 \end{pmatrix} = x\,\mathbf{1} + P,$$

where $\mathbf{1}$ and P are the identity and permutation on $\mathbb{C}^2 \otimes \mathbb{C}^2$, respectively. In this notation

$$p_{\Delta=1}(x) = -i \log \frac{x+1}{x}, \qquad \varepsilon_{\Delta=1}(x) = \frac{2}{x(x+1)},$$

so magnons with $p_k = 0$ correspond to (additive) Bethe roots x_k running away to infinity. (One sometimes further sets $\lambda = x + 1/2$, but we won't need that.) To determine the leading behaviour of the B operator from the algebraic Bethe ansatz as $x \to \infty$ observe that the horizontal boundary conditions in

$B(z) = z$ 

force us to pick up the subleading term P in the L-operator $x\,\mathbf{1} + P$ at least once: the incoming horizontal blue line has to turn up. Therefore

$$\lim_{q\to 1} \frac{B(q^x)}{(q-q^{-1})^L} = x^{L-1} \sum_{j=1}^{L} \langle \cdots\cdots | \, P_{0j} \, | \text{——} \rangle + l.d. = x^{L-1}\,\Sigma^- + l.d.,$$

where '$l.d.$' denotes terms of lower degree in x. Here we expressed the permutation as $P = \sigma_0^+\sigma_j^- + \sigma_0^-\sigma_j^+ + (\sigma_0^z\sigma_j^z + 1)/2$, cf. $\check{R}'(1)$ in exercise 1.3, and used $\langle \cdots | \sigma_0^+ | — \rangle = 1$ while $\langle \cdots | \sigma_0^- | — \rangle = \langle \cdots | \sigma_0^z | — \rangle = \langle \cdots | \mathbf{1} | — \rangle = 0$. The conclusion is that in the isotropic limit the B-operators act as the $\mathfrak{sl}_2$ lowering operators, up to a rescaling to keep everything finite, when $x_k \to \infty$, as we claimed. (For general q the leading behaviour of B gives rise to the $U_q(\mathfrak{sl}_2)$-lowering operator instead.)

2.5. **Yang–Baxter algebra representations and inhomogeneous monodromy matrix.** Consider the Yang–Baxter bialgebra $\mathcal{A}$ with generators $(\hat{\mathcal{T}}_i^j(z))_{i,j=1,\ldots,n}$ associated to an R-matrix $R(z) = (R_{ik}^{j\ell}(z))_{i,j,k,\ell=1,\ldots,n}$ satisfying the Yang–Baxter equation.

- *Show that* $\hat{\mathcal{T}}_i^j(z) \overset{\rho_z}{\longmapsto} (R_{ik}^{j\ell}(z/w))_{k,\ell=1,\ldots,n}$ *defines a representation of* $\mathcal{A}$ *on* $\mathbb{C}^n$.
  One needs to check that the RTT relations are satisfied. But with this particular representation, the RTT relations are nothing but the Yang–Baxter equation for R.
- *Show that* $\hat{\mathcal{T}}_i^j(z) \mapsto \mathcal{T}_i^j(z; z_1, \ldots, z_L)$ *(inhomogeneous transfer matrix) is the tensor product representation* $\mathbb{C}^n(z_1) \otimes \mathbb{C}^n(z_2) \otimes \cdots \otimes \mathbb{C}^n(z_L)$ *of* $\mathcal{A}$.
  Let us consider $L = 2$. Then the representation $\mathbb{C}^n(z_1) \otimes \mathbb{C}^n(z_2)$ is by definition given by

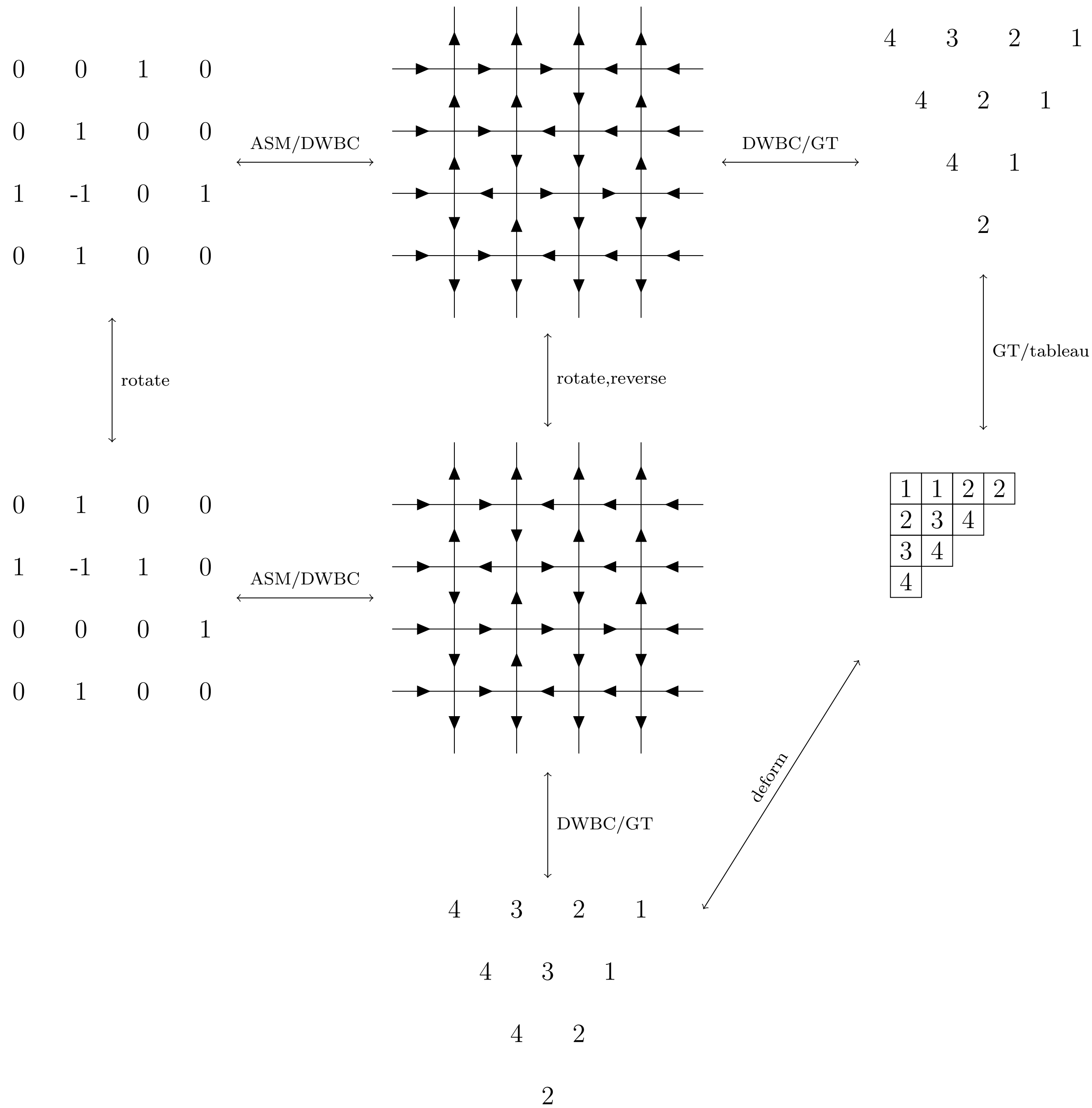
$$(\rho_{z_1} \otimes \rho_{z_2})\Delta(\hat{\mathcal{T}}_i^j(z)) = \sum_{k=1}^{n} (\rho_{z_1} \otimes \rho_{z_2})(\hat{\mathcal{T}}_i^k(z) \otimes \hat{\mathcal{T}}_k^j(z))$$

Acting on a basis vector:

$$\begin{aligned}(\rho_{z_1} \otimes \rho_{z_2})\Delta(\hat{\mathcal{T}}_i^j(z))e_{a_1} \otimes e_{a_2} &= \sum_{k,b_1,b_2=1}^{n} R_{ia_1}^{kb_1}(z/z_1) R_{ka_2}^{jb_2}(z/z_2) e_{b_1} \otimes e_{b_2} \\ &= \sum_{b_1,b_2=1}^{n} (R_{02}(z/z_2) R_{01}(z/z_1))_{ia_1a_2}^{jb_1b_2} e_{b_1} \otimes e_{b_2}\end{aligned}$$

which is the desired expression.
  Inductively, one obtains the result for any L.

3.1. **Alternating Sign Matrices, tableaux and Gelfand–Tseytlin patterns.** We provide here the image of the working example under the various bijections. The proofs of the statements in the general case are straightforward but tedious!



3.2. **NilHecke solution of Yang–Baxter equation and Schubert polynomials.**

- *Show that R satisfies the Yang–Baxter equation with additive spectral parameters.*
  The Yang-Baxter equation for $R(x)$ is equivalent to the following braid-like equation for $\check{R}(x) := PR(x)$ ($Px \otimes y = y \otimes x$):

$$\check{R}_{12}(x)\check{R}_{23}(x+y)\check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(x+y)\check{R}_{23}(x)$$

(where subscripts denote which spaces they act on in $(\mathbb{C}^r)^{\otimes 3}$).

In our case, $\check{R}(x)$ can be written

$$\check{R}(x) = 1 + xE, \qquad E^{\ell j}_{ik} = \begin{cases} 1 & i = j < k = \ell \\ 0 & \text{else} \end{cases}, \qquad 1 \le i, j, k, \ell \le r$$

E satisfies the two properties:

(1) $E^2 = 0$. Indeed, $(E^2)^{ab}_{ef} = \sum_{c,d=1}^{r} E^{ab}_{cd} E^{cd}_{ef}$, and for this to be nonzero, one has to satisfy both inequalities $c < d$ and $d < c$ which is impossible.

(2) The braid relations:

$$E_{12}E_{23}E_{12} = E_{23}E_{12}E_{23}$$

(where again subscripts denote which spaces they act on in $(\mathbb{C}^r)^{\otimes 3}$). Indeed, one proves easily the formula

$$(E_{12}E_{23}E_{12})^{abc}_{def} = (E_{23}E_{12}E_{23})^{abc}_{def} = \begin{cases} 1 & c = d < b = e < a = f \\ 0 & \text{else} \end{cases}$$

(Thus, the $E_{i\,i+1}$ generate the so-called nilHecke algebra.)

We now compute

$$\begin{aligned} \check{R}_{12}(x)\check{R}_{23}(x+y)\check{R}_{12}(y) &= (1 + xE_{12})(1 + (x+y)E_{23})(1 + yE_{12}) \\ &= 1 + (x+y)(E_{12} + E_{23}) + x(x+y)E_{12}E_{23} + y(x+y)E_{23}E_{12} \\ &\quad + xyE_{12}^2 + xy(x+y)E_{12}E_{23}E_{12} \end{aligned}$$

and similarly

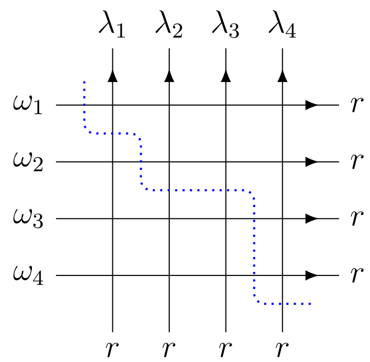
$$\begin{aligned} \check{R}_{23}(y)\check{R}_{12}(x+y)\check{R}_{23}(x) &= (1 + yE_{23})(1 + (x+y)E_{12})(1 + xE_{23}) \\ &= 1 + (x+y)(E_{12} + E_{23}) + x(x+y)E_{12}E_{23} + y(x+y)E_{23}E_{12} \\ &\quad + xyE_{23}^2 + xy(x+y)E_{23}E_{12}E_{23} \end{aligned}$$

Using the two properties above show the equality.

- At $r = 2$, we recognize the R-matrix of the *rational 5-vertex model*. The partition function becomes the partial DWBC partition function (or offshell Bethe state of the model), and if we set all ys equal zero, we conclude that S_λ is a Schur polynomial.
- *For general r, show that S_λ is a* homogeneous polynomial *in the xs and ys. What is its degree?*

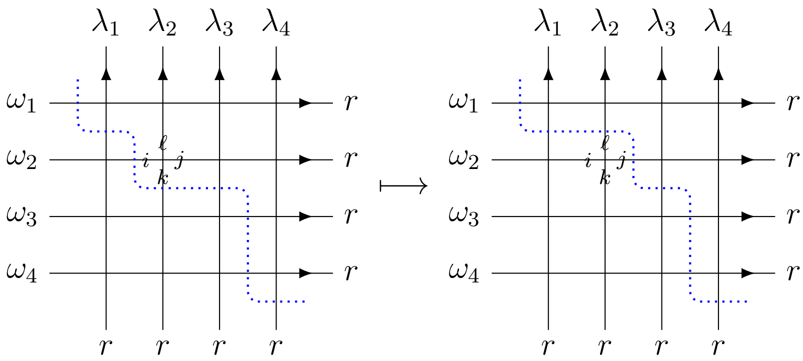
Let us define the *inversion number* of a k-tuple $(\alpha_1, \ldots, \alpha_k)$ to be the number of $i < j$ such that $\alpha_i > \alpha_j$.

Now to any South-West "slice" of a configuration contributing to the partition function let us associate the inversion number of the labels it traverses:



which we read North-West to South-East.

The lowest possible such path traverses $(\omega_1, \dots, \omega_n, r, \dots, r)$ which forms a weakly increasing sequence, thus has zero inversion number. The highest possible such path traverses $(\lambda_1, \dots, \lambda_n, r, \dots, r)$, and thus has the inversion number of λ itself. Finally, shifting the path by making it go across one more crossing:



leads to two possibilities: either the sequence of labels is the same (i.e., $i = \ell, k = j$), in which case the crossing in question has Boltzmann weight 1; or the two labels i and k get swapped (i.e., $i = j$, $k = \ell$), which can only happen if $i < k$ (and thus, $\ell > j$), in which case the Boltzmann weight is $x_i - y_j$. We conclude that the degree of that Boltzmann weight is exactly the difference of inversion numbers of the two paths.

Putting everything together, we find the degree of the weight of any configuration is given by the inversion number of λ. Thus, S_λ is homogeneous of that degree.

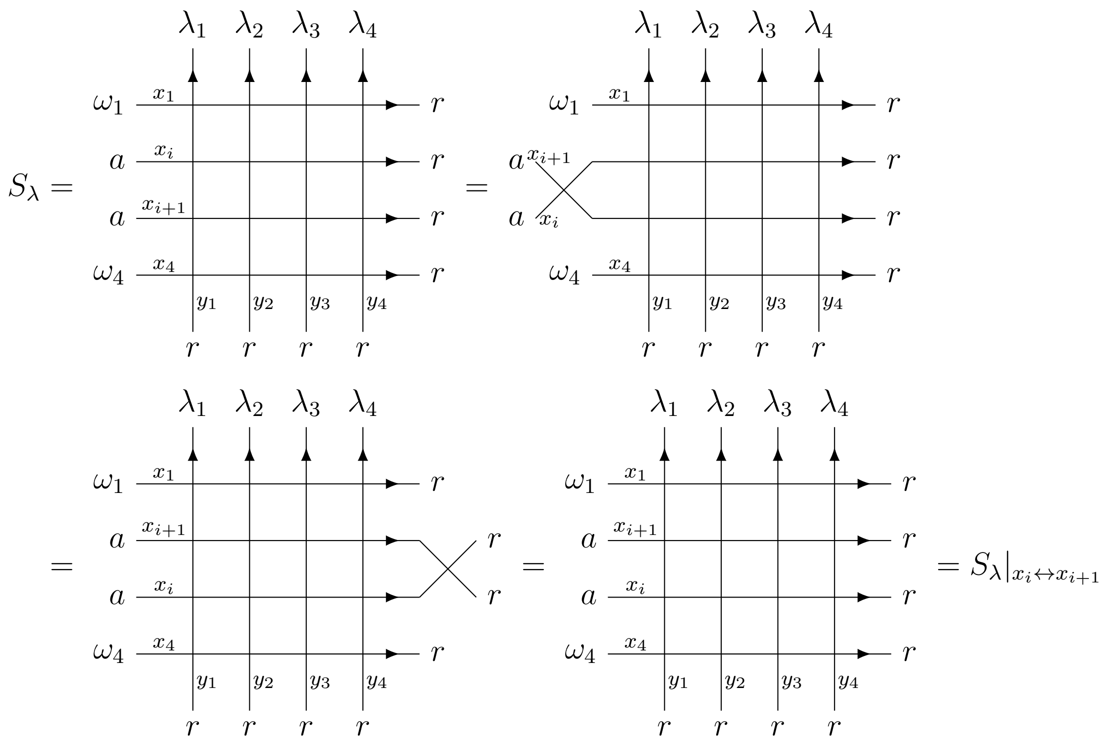
Denote $I_a = \{i : \omega_i = a\}$ for $a = 1, \dots, r$.

- *Show that S_λ does not depend on the x_i, $i \in I_r$.*

Say $I_r = \{m, \dots, n\}$. Then the bottom $n-m+1$ rows of the partition function are "frozen" in the sense that all the labels can only take the value r, and in particular have Boltzmann weights in that region are 1. This means that S_λ is independent of $x_m, \dots, x_n$.

- *Show that for each $a \in \{1, \dots, r\}$, S_λ is invariant by permutation of the x_i, $i \in I_a$.*

This is the usual "unzipping" argument. Suppose $\omega_i = \omega_{i+1} = a$. Then

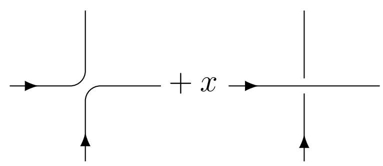


Since any permutation is a product of elementary transpositions, we get the desired result.

- *Show that $S_\lambda = S_\mu$ where μ is the standardization of λ.* Consider the mapping $i \mapsto \omega_i = \lambda_{\mu^{-1}(i)}$. Applying it to all labels of a configuration contributing to S_μ, one obtains a configuration contributing to S_λ, and it's not hard to see that it provides a bijection between the former and the latter, implying $S_\mu = S_\lambda$.
- *Show that if $\underline{\lambda}$ is weakly decreasing, then*

$$S_\lambda = \prod_{j=1}^{n} \prod_{i=1}^{\lambda_j} (x_i - y_j)$$

The statement amounts to saying that there's a unique configuration with nonzero Boltzmann weight contributing to S_λ. This is true by inspection, building the configuration from top to bottom. On an example, using the convenient graphical way to depict the two terms of the R-matrix

$R(x) =$ 

where the second term only occurs if labels are ordered, we have

$$\lambda = (4,2,1,3),\ \ \underline{\lambda} = (3,1,0,0),$$